Classical Mechanics
Day 3

__Outline__

I Homogeneous linear differential equations

II Last time & damped oscillator wrap up

III Driven damped oscillator

$$a_n \frac{d^n x}{dt^n} + \cdots + a_1 \frac{dx}{dt} + a_0 x = 0.$$

Any eqn. like this can be solved by x of the form:

$$x(t) = e^{rt}$$

w/ r const. (our standard guess).

__why it works__: $x = e^{rt} \Rightarrow \frac{dx}{dt} = re^{rt}$

$\frac{d^2x}{dt^2} = r^2 e^{rt}$, ..., $\frac{d^n x}{dt^n} = r^n e^{rt}$

I Homog. linear eqn.s w/ constant coeff.:

__Linear__: $a_n(t) \frac{d^n x}{dt^n} + a_{n-1}(t) \frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_1(t) \frac{dx}{dt} + a_0(t) x = f(t)$

__Homogeneous__: $f(t) = 0$.

__Constant coeff.__: $a_0, a_1, \ldots, a_n$ do __not__ depend on t.

So,

$$a_n r^n e^{rt} + a_{n-1} r^{n-1} e^{rt} + \cdots + a_0 e^{rt} = 0$$

$$\Rightarrow a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0$$

("roots")

Fundamental thm. of Algebra guarantees that there are n soln.s to Eq. (root).

__clean up__: Collect the a_i and the deriv.s into a __differential__

<u>Operator:</u>

$$D \equiv a_n \frac{d^n}{dt^n} + a_{n-1}\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1 \frac{d}{dt} + a_0$$

<u>Ex.</u>: damped oscillator

$$D \equiv \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$$

$$D x = \ddot{x} + 2\beta\dot{x} + \omega_0^2 x$$

This operator is <u>linear</u>

$$D(ax) = aDx\,,\quad D(x_1+x_2) = Dx_1 + Dx_2$$

with the r_i solutions of Eq. (roots).

<u>Except</u>: if the algebraic eqn. gives multiple roots, then I'm missing solutions. S'pose

$$\left(\frac{d}{dt}-r\right)\left(\frac{d}{dt}-r\right)x = 0$$

Let $y = \left(\frac{d}{dt}-r\right)x$ and so

$$\left(\frac{d}{dt}-r\right)y = 0$$

$$\Rightarrow \quad y = Ce^{rt}$$

For linear & homog. diff. eqn.s any linear combo of solns is a soln because

$$D(ax_1 + bx_2) = aDx_1 + bDx_2$$
$$= 0 + 0 = 0.\ \checkmark$$

For n^{th} order eqn.s the linear combo w/ n constants is the general solution

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \cdots + C_n e^{r_n t}$$

But, then

$$\left(\frac{d}{dt}-r\right)x = Ce^{rt}$$

Check $x = Ct\,e^{rt}$ solves it

$$\left(Ce^{rt} + C\cancel{t}re^{rt} - rC\cancel{t}e^{rt}\right) \stackrel{?}{=} Ce^{rt}$$

yes! In general, for an m times repeated root, soln.s are

$$e^{rt},\ te^{rt},\ t^2e^{rt},\ \ldots,\ t^{m-1}e^{rt}.$$

Last time: damped oscillator

$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

w/ $r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$, $r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$

$\beta = \frac{b}{2m}$ ← damping coeff., $\omega_0 = \sqrt{\frac{k}{m}}$, $B \equiv \sqrt{\beta^2 - \omega_0^2}$

$\omega_1 \equiv \sqrt{\omega_0^2 - \beta^2}$.

Started to explore three regimes

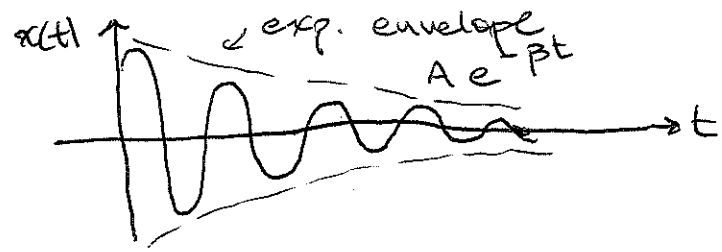
(1) Overdamped: $\beta > \omega_0$. Exponentially damped motion with two characteristic times

$$\tau_1 = \frac{1}{\beta - B}, \quad \tau_2 = \frac{1}{\beta + B}$$

(2) Underdamped: $\omega_0 > \beta$

showed that

$$x(t) = A e^{-\beta t} \cos(\omega_1 t - \delta)$$

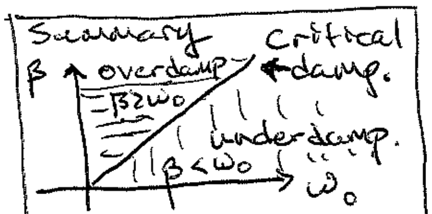


Also two times at work here:

characteristic damping time $\tau_1 = 1/\beta$;

and period of oscillation

$$\tau_2 = \frac{2\pi}{\omega_1}$$



Question: How many oscillations in one characteristic time?

$$\# = \frac{\tau_1}{\tau_2} = \frac{1}{\beta \frac{2\pi}{\omega_1}} = \frac{1}{\pi} \frac{\omega_1}{2\beta} \approx \frac{1}{\pi} \frac{\omega_0}{2\beta}$$

small damping or large mass

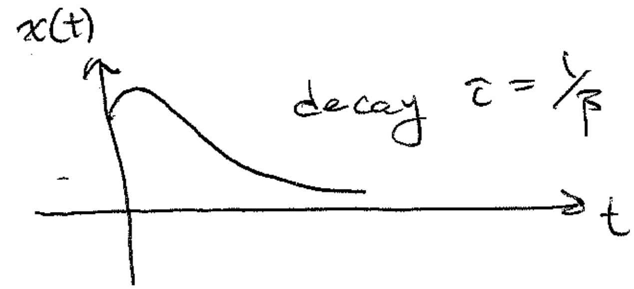
quality factor $Q \equiv \frac{\omega_0}{2\beta}$

(3) Critical damping: $\beta = \omega_0$

$\Rightarrow r_1 = r_2$ multiple root

General solution

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t}$$
$$= e^{-\beta t}(C_1 + C_2 t)$$



III <u>Driven damped</u> oscillator

Think of a child on a swing: oscillator (swing), damping (friction) driving (you pushing the child)

$$F_{net} = -kx - bv + F(t)$$

In general the driving depends on time.

So, $m\ddot{x} + b\dot{x} + kx = F(t)$

$$\Rightarrow \quad \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) \equiv \frac{F(t)}{m} \quad \text{(drive)}$$

ω_0 the "natural freq." and

ω the "driving freq.".

This is an example of an inhomogeneous linear diff. eqn.

S'pose $x_h(t)$ is the general soln. to the associated homog. eqn then

$$\overset{\text{diff. op.}}{D}\, x_h = 0$$

Small f is the force per unit mass. We'll be particularly interested in

$$f(t) = f_0 \cos(\omega t)$$

('cause it's physically reasonable <u>and</u> soluable — can build more complicated driving out of cosines too). Note well the distinction btwn

$$D \equiv \frac{d^2}{dt^2} + 2\beta\frac{d}{dt} + \omega_0^2$$

S'pose further that $x_p(t)$ is any old ("particular") soln. to the inhomog. eqn:

$$D\,x_p(t) = f(t)$$

Then $x_h + x_p$ solves the inhomog. eqn.

$$D(x_h + x_p) = Dx_h + Dx_p = 0 + f(t)$$

and is, in fact, the general soln.