Mechanics Day 5

Today

I Last time

II Wrap up Resonance

III Motivations for the Calculus of Variations

IV The Chain Rule

I We found the particular solution

$$z_p = C e^{i\omega t} = \overset{\text{real}}{A} e^{-i\delta} e^{i\omega t}$$

to the complex damped (harmonically) driven oscillator equation

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = f_0 e^{i\omega t}.$$

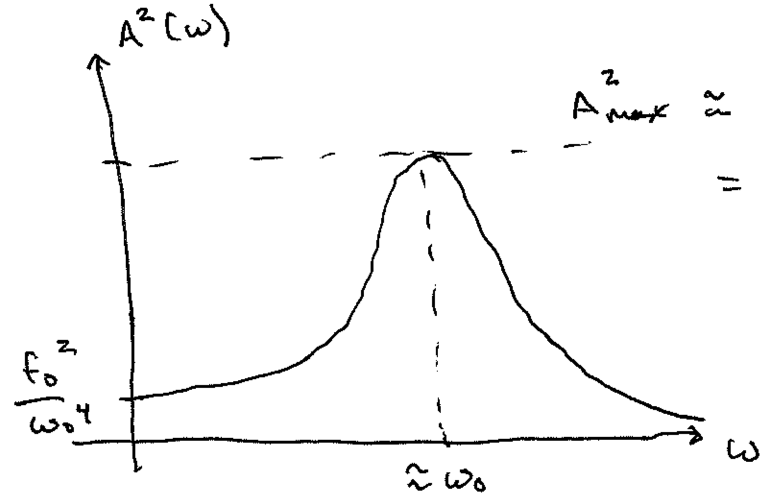
We simplified the amplitude A and the phase δ to find

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2}}$$

and

$$\tan\delta = \frac{2\beta\omega}{\omega_0^2 - \omega^2}.$$

We plotted the amplitude as a function of the drive frequency and found



$$A^2_{max} \approx A^2(\omega_0) = \frac{f_0^2}{4\beta^2\omega_0^2}$$

We characterized the max of the resonance.

Then,

$$A_{max}^2 \approx A^2(\omega_0) = \frac{f_0^2}{4\beta^2\omega_0^2}$$

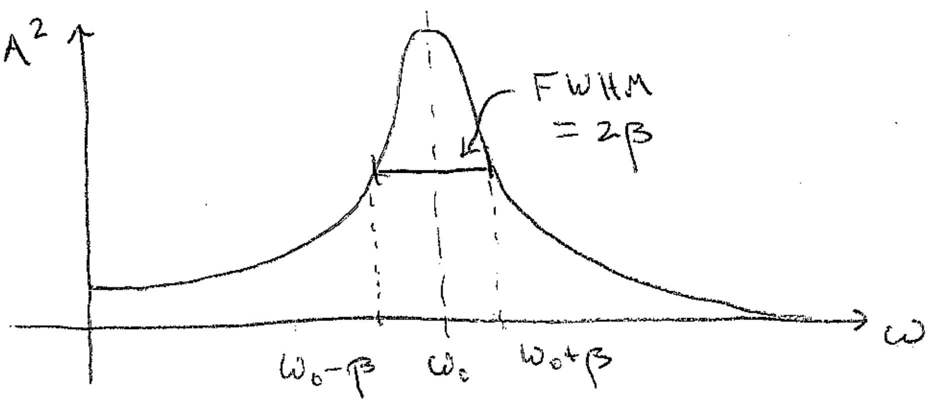
We've characterized the height of the resonance. What about its width? This is done with:

FWHM = "Full width at Half Max"

or

HWHM = "Half Width at Half Max"

Again assume ($\beta \ll \omega_0$) β is small



A dimensionless measure of the sharpness of the resonance is the Quality factor

$$Q = \frac{\omega_0}{2\beta}$$

Large value of $Q \Rightarrow$ narrow resonance.

Then for $\omega = \omega_0 \pm \beta$,

$$\omega_0^2 - \omega^2 = (\omega_0 + \omega)(\omega_0 - \omega) \approx 2\omega_0(\mp\beta)$$

and

$$4\beta^2\omega^2 \approx 4\beta^2\omega_0^2 + O(\beta^3)$$

So,

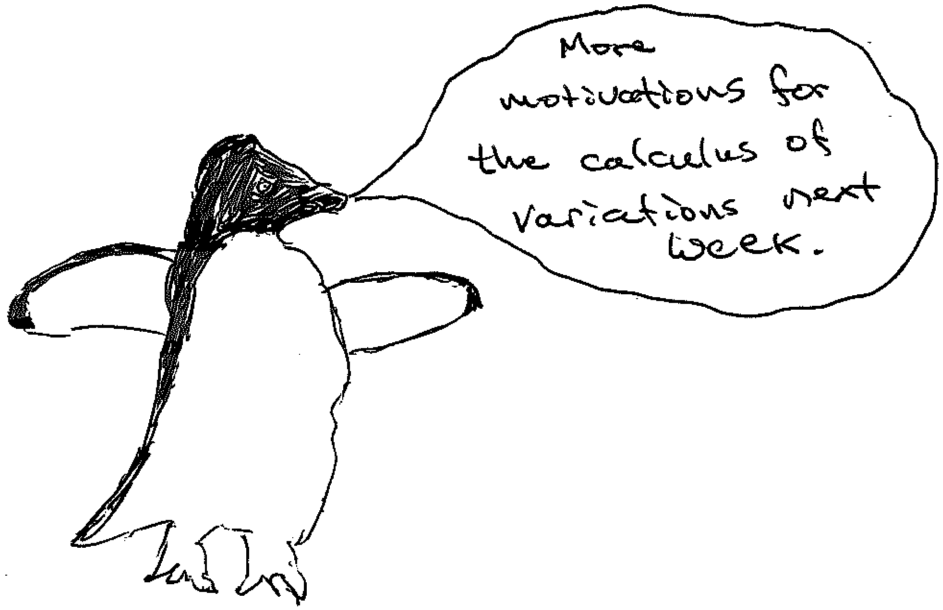
$$A^2(\omega_0 \pm \beta) \approx \frac{f_0^2}{(\mp 2\beta\omega_0)^2 + 4\beta^2\omega_0^2} = \frac{f_0^2}{8\beta^2\omega_0^2} \approx \frac{1}{2}A_{max}^2$$

Examples of resonance:

Electrical: Many, many, but e.g. a radio tuner.

Optical spectra: e.g. the absorption spectrum of the sun,

Nuclear: NMR = Nuclear Magnetic Resonance — good for medical imaging.



III. New approach to Mechanics called a variational principle. Based on new mathematics: the calculus of variations.

Because this approach is new and its tools are new, it can be easy to lose the forest for the trees. This motivation will try to mitigate this.

Recall that Newton's laws are a set of axioms:

I. Velocity is const, unless body is acted on by a force.

II. The acceleration is given by

$$\vec{a} = \vec{F}/m$$

III Mutual forces of action and reaction are equal, opposite and collinear.

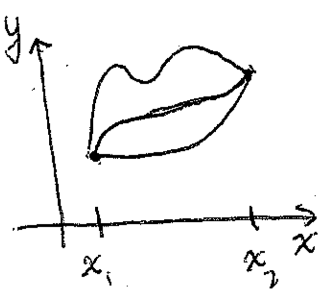
They have been promoted to the status of laws because of their incredible predictive success.

But, <u>other</u> axioms are possible.

A variational principle expresses the idea that the correct motion of a system can be predicted by extremizing (minimizing, maximizing, saddling) an integral:

e.g. $I = \int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx$

Eventually the integral itself will take on physical meaning (particularly in Quantum Mechanics), but for now we will focus on the path $y(x)$ that we feed to the integral. By changing this path we change the value of I and our immediate goal will be to find a condition on the path that guarantees it will extremize I.



Different paths give different integrals

Physically our independent variable will be t and our path will be, e.g. $x(t)$. We will find an integral $S[x(t)]$ that is extremized when $x(t)$ satisfies the physical E.O.M. This is our goal.

"equations of motion"

Strategy: Math first then physics.

Why?: Allows us to separate the calculation and its interpretation.

Notation: I for integral
x for independent variable
y for dependent variable } Appropriate for math
$y(x)$ for the path

We will adopt physical notation starting next week.

Why bother with all of this?

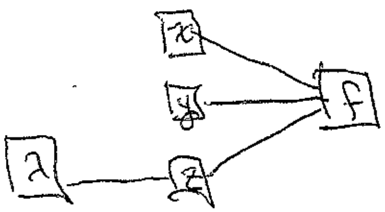
Many reasons, but one important one is that it will free us from the shackles of Cartesian coord.s. Adapt our coord.s to the symmetry of the system. I'll list more as we go.

Today: Euler's method (not in your book)
Friday: Lagrange's method (in your book)

IV Consider $f = f(x, y, z)$ and the case in which $z = z(\lambda)$, so that

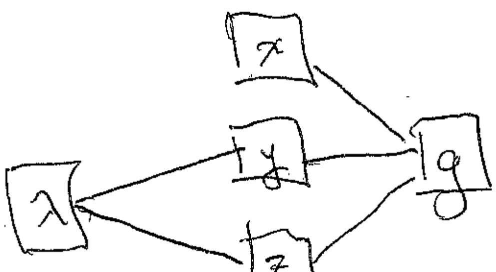
$$f = f(x, y, z(\lambda))$$

Schematically



If we shake (that is, vary) y then $\frac{\partial f}{\partial y}$

Two subtleties (1) What if $g = g(x, y(\lambda), z(\lambda))$?

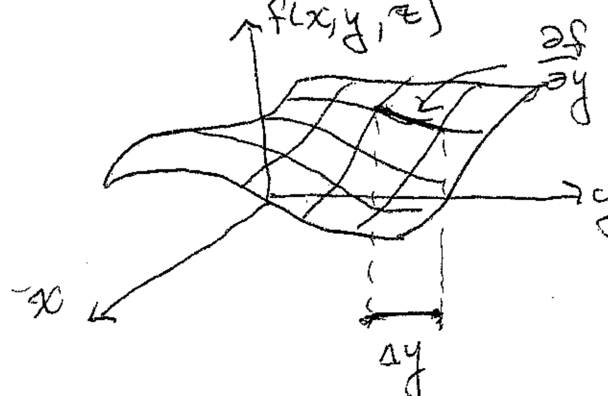


$$\frac{d}{d\lambda}(g) = \frac{\partial g}{\partial y}\frac{dy}{d\lambda} + \frac{\partial g}{\partial z}\frac{dz}{d\lambda}$$

← you add contributions

If $\frac{dy}{d\lambda} = 0$ recovers previous example

gives how f changes when you vary y keeping all other variables fixed, graphically



$\frac{\partial f}{\partial y}$ = slope of tangent line

What about if you shake λ? Causes z to shake and $\frac{df}{d\lambda} = \frac{\partial f}{\partial z}\frac{dz}{d\lambda}$

(2) What if $f = f(x, y, z(\lambda))$ and we know $z(\lambda) = \lambda^2$?

You might like to write

$$f = f(x, y, z(\lambda)) = f(x, y, \lambda^2)$$

But $\frac{d(f)}{d\lambda} = \frac{\partial f}{\partial \lambda^2}\frac{d\lambda^2}{d\lambda}$ (Confusing Eqn — Don't use it!)

Instead $\frac{d(f)}{d\lambda} = \frac{\partial f}{\partial z}\frac{dz}{d\lambda} = \frac{\partial f}{\partial z}\cdot(2\lambda)$.

Sometimes we write $D_3 f(x, y, z) \equiv \frac{\partial f}{\partial z}$ to avoid this confusion.