Mechanics
Day 6

Today

I Last time

II Euler's Method in the calculus of variations

III An example of Euler's method

I. • Found the relation between the full width at half max (FWHM) and the quality factor

$$Q = \frac{\omega_0}{\text{FWHM}}.$$

• Extensively motivated the calculus of variations. E.g. can find E.O.M. in any coordinates.

• Reviewed the chain rule.

It doesn't matter what you call the slots of a function, the partial derivative w.r.t. a particular slot is always well-defined. For $f = f(x, y, z)$, the notations

$$\frac{\partial f}{\partial y} = D_2 f = \partial_y f = f_{,y}$$

all mean the same thing.

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II Euler's Method

S'pose that the curve in Figure 1 extremizes the integral

$$I = \int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx$$

We want to find an equation that determines this curve $y(x)$.

We will proceed in an approximate manner, similar to a Riemann sum in calculus:

(1) Divide the interval btwn $x = x_1$ and $x = x_2$ into many subintervals of width Δx.

(2) Approximate the integral by a sum

$$I = \int_{x_1}^{x_2} f(x, y, y')\, dx \approx \sum_{x=x_1}^{x_2} f(x, y, y')\, \Delta x$$

In each term of this sum evaluate f at the initial pt, e.g. x_N, $y(x_N) = y_n$ for interval $[x_N, x_O]$.

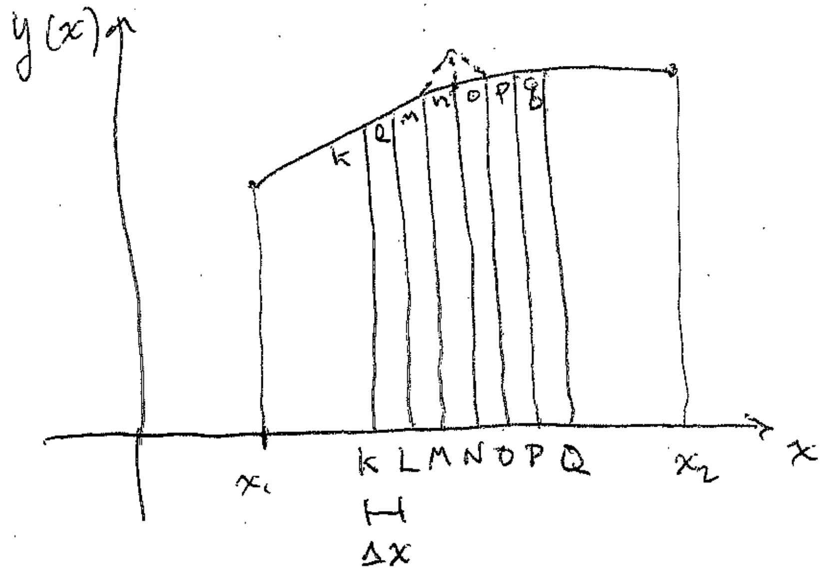


Figure 1

(3) Approximate the derivative $y' = \frac{dy}{dx}$ by the slope of the straight line connecting the end pts of the interval. That is,

$$y'_m \equiv \frac{\Delta y}{\Delta x} = \frac{y_n - y_m}{\Delta x} \simeq y'(x_M)$$

All of these approximations become excellent in the limit of many subintervals, ie. as $x \to 0$.

Now, let's change y_n (see dashed lines in Fig. 1). This changes the sum, but only the terms involving y_n. The sum no longer depends on the whole path just the values of f at the discrete pts, x_n, y_n and y_n'. So, we can extremize it with regular Calculus:

$$\frac{\partial}{\partial y_n}\left(\sum_{x=x_1}^{x_2} f(x,y,y')\,\Delta x\right) = 0.$$

Setting this equal to zero and rearranging gives,

$$\frac{\partial f}{\partial y_n}\cdot\Delta x - \left(\frac{\partial f}{\partial y_n'} - \frac{\partial f}{\partial y_m'}\right) = 0$$

Dividing by Δx,

$$\frac{\partial f}{\partial y_n} - \frac{1}{\Delta x}\left(\frac{\partial f}{\partial y_n'} - \frac{\partial f}{\partial y_m'}\right) = 0$$

In the limit $\Delta x \to 0$ this is

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0}$$

The Euler-Lagrange Equation

Let's do it:

$$\frac{\partial}{\partial y_n}\left(\sum_{x=x_1}^{x_2} f(x,y,y')\,\Delta x\right)$$

terms not involving y_n

$$= \frac{\partial}{\partial y_n}\left(\cdots + f\left(x_M, y_m, \frac{y_n - y_m}{\Delta x}\right)\Delta x + f\left(x_N, y_n, \frac{y_o - y_n}{\Delta x}\right)\Delta x + \cdots\right)$$

chain rule

$$= \frac{\partial f}{\partial y_m'}\frac{\partial}{\partial y_n}\left(\frac{y_n - y_m}{\Delta x}\right)\cdot\Delta x + \frac{\partial f}{\partial y_n}\cdot\Delta x + \frac{\partial f}{\partial y_n'}\frac{\partial}{\partial y_n}\left(\frac{y_o - y_n}{\Delta x}\right)\cdot\Delta x$$

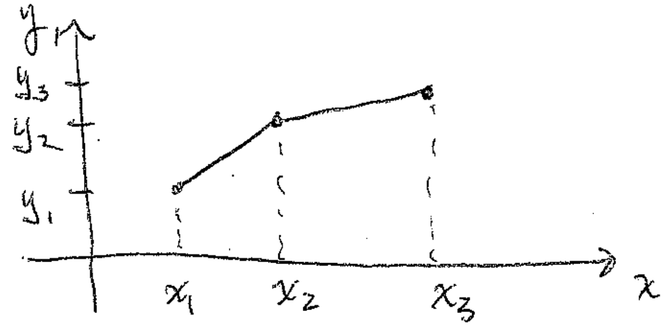
N.B.: Important distinction btwn partial and total derivatives here.

Euler's method is nice bcse it is closely tied to the geometry. Next lecture we will look at Lagrange's method, which is mathematically slicker.

II Let's do an example to illustrate this.

Q: What is the shortest path connecting two points in the xy-plane?

Focus first on (x_1, y_1) and (x_3, y_3)



So we impose,

$$\frac{\partial L}{\partial y_2} = 0$$

$$\Rightarrow \frac{1}{2}\frac{1}{\sqrt{\Delta x^2 + (y_2-y_1)^2}} \cdot 2(y_2-y_1) + \frac{1}{2}\frac{1}{\sqrt{\Delta x^2 + (y_3-y_2)^2}} \cdot 2(y_3-y_2)(-1) = 0$$

$$\Rightarrow \frac{y_2-y_1}{\sqrt{\Delta x^2 + (y_2-y_1)^2}} = \frac{y_3-y_2}{\sqrt{\Delta x^2 + (y_2-y_1)^2}}$$

Factor out Δx^2 to find

The length of the path connecting them depends on our choice of (x_2, y_2)

This length is

$$L = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} + \sqrt{(x_3-x_2)^2 + (y_3-y_2)^2}$$

$$= \sqrt{\Delta x^2 + (y_2-y_1)^2} + \sqrt{\Delta x^2 + (y_3-y_2)^2}$$

We want to choose y_2 such that L is a minimum.

$$\frac{(y_2-y_1)/\Delta x}{\sqrt{1 - \frac{(y_2-y_1)^2}{\Delta x^2}}} = \frac{(y_3-y_2)/\Delta x}{\sqrt{1 - \frac{(y_3-y_2)^3}{\Delta x^2}}}$$

and we get equality if

$$\left(\frac{y_2-y_1}{\Delta x}\right) = \left(\frac{y_3-y_2}{\Delta x}\right)$$

that is, if the slope is constant!

It's a straight line!!