Mechanics
Day 7

Today

I Last time

II Lagrange's method in the calculus of variations

III Example: Geodesics on the cylinder

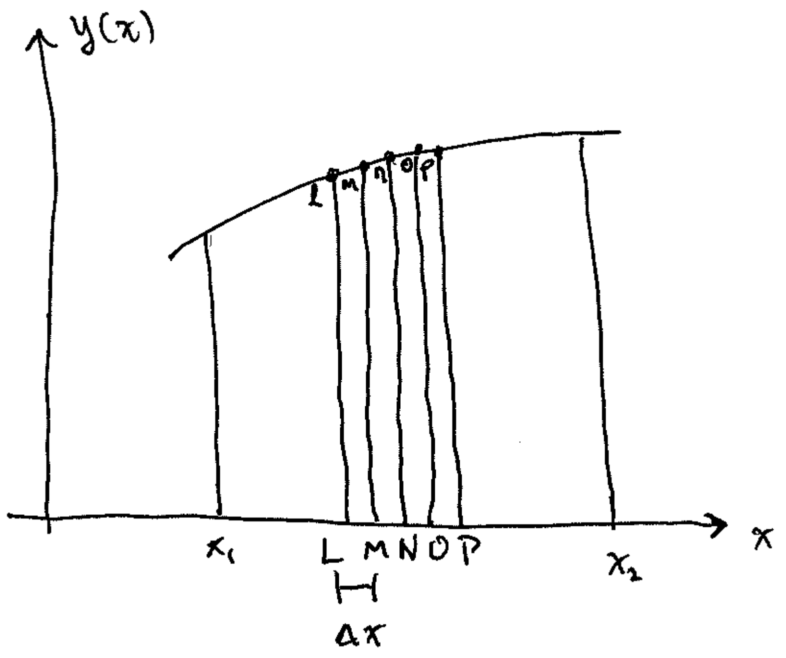


Figure 1

I. We found

$$\frac{\partial f}{\partial y_n} - \frac{1}{\Delta x}\left(\frac{\partial f}{\partial y'_n} - \frac{\partial f}{\partial y'_m}\right) = 0$$

as a condition for

$$\sum_{x=x_1}^{x_2} f(x, y, y')\,\Delta x$$

to be extremized w.r.t. variations of y_n. See Fig 1 for notation.

In the limit $\Delta x \to 0$ we have

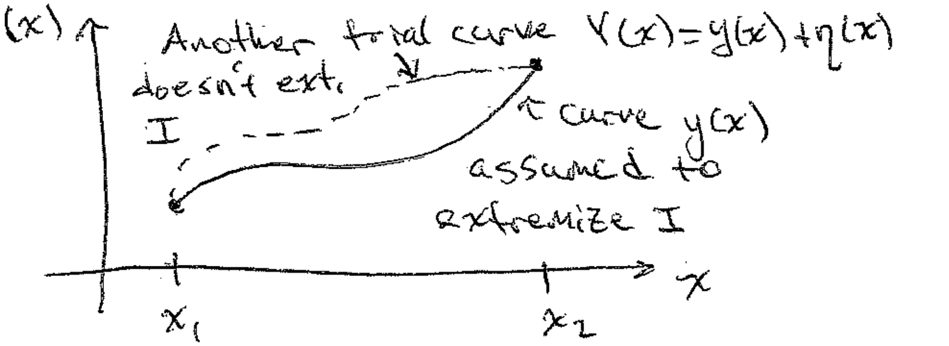
$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

Notice: Our derivation holds for any point on the curve. That is, an arbitrary x and $y(x)$, hence all of them. Thus, in using Euler's method you can always focus on a small segment of the curve.

II Lagrange's Method

We want to find the curve $y(x)$ that extremizes

$$I = \int_{x_1}^{x_2} f(x, y(x), y'(x))\, dx$$



normal function of α, $I(\alpha)$.
So the condition that I is extremized is just

$$\frac{dI}{d\alpha} = 0$$

Subtlety: We assume $y(x)$ is the "right" curve, so that we know where to evaluate, namely

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = 0. \quad \perp$$

Lagrange also wants to turn this into a calculus problem. He found a clever way to do it:
Write your trial path as

$$Y(x) = y(x) + \alpha\,\eta(x)$$

in terms of a parameter α.
Now, you can turn on a variation of the whole curve just by changing α. In particular, I becomes a

Let's do it, if $Y(x) = y(x) + \alpha\,\eta(x)$
then

$$Y'(x) = y'(x) + \alpha\,\eta'(x)$$

so, if

$$I(\alpha) = \int_{x_1}^{x_2} f(x, Y, Y')\, dx$$
$$= \int_{x_1}^{x_2} f(x, y(x) + \alpha\eta, y' + \alpha\eta')\, dx$$

Then, taking deriv. inside integral makes it a partial:

$$\frac{dI}{d\alpha} = \frac{d}{d\alpha}\int_{x_1}^{x_2} f\, dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial\alpha}\, dx$$

Now, $\frac{\partial}{\partial\alpha}\left[f(x, y+\alpha\eta, y'+\alpha\eta')\right]$

chain rule

$$= \frac{\partial f}{\partial y}\cdot\frac{\partial}{\partial\alpha}(y+\alpha\eta) + \frac{\partial f}{\partial y'}\cdot\frac{\partial}{\partial\alpha}(y'+\alpha\eta')$$

$$= \frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'$$

So,

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\eta + \frac{\partial f}{\partial y'}\eta'\right)dx$$

"Can switch derivative at the cost of a minus sign and a boundary term"

Then,

$$\int_{x_1}^{x_2}\frac{\partial f}{\partial y'}\frac{d\eta}{dx}\,dx = \underbrace{\left(\frac{\partial f}{\partial y'}\eta\right)\Big|_{x_1}^{x_2}}_{\substack{=0 \text{ bcse of assumed} \\ \text{b.c. on } \eta}} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\eta\,dx$$

and

$$\frac{dI}{d\alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right)\eta\,dx$$

To simplify the 2nd term use integration by parts

$$\int_{x_1}^{x_2}\frac{d}{dx}\left(f(x)g(x)\right)dx = f(x)g(x)\Big|_{x_1}^{x_2}$$

(integral = antiderivative), But, also

$$\int_{x_1}^{x_2}\frac{d}{dx}(fg)\,dx = \int_{x_1}^{x_2}f'g\,dx + \int_{x_1}^{x_2}fg'\,dx$$

Putting these together

$$\int_{x_1}^{x_2}\frac{df}{dx}g\,dx = f(x)g(x)\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2}f\frac{dg}{dx}\,dx$$

Want to set $\frac{dI}{d\alpha} = 0$.

The pinnacle of the α cleverness is that this is to hold for <u>all</u> η. So, what can we say when

$$\int_{x_1}^{x_2}g(x)\eta(x)\,dx = 0$$

for all η? S'pose $g \neq 0$ and choose η positive whenever g is and similarly for negative values.

Then $g(x)\cdot\eta(x) > 0$ and
$\uparrow$ always!

$$\int_{x_1}^{x_2} g(x)\eta(x)\,dx \neq 0,$$

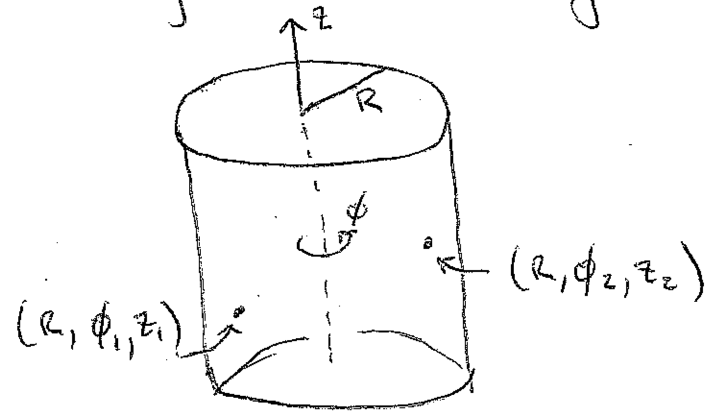
a contradiction! $\Rightarrow \boxed{g(x)=0}$.

So,

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \text{E-L eqns again!}}$$

That completes our 2nd proof.
Can go to examples (at last!).

III. Find the geodesics on a cylinder



Q: Shortest path connecting these pts?
Describe as $\phi(z)$. Arclength

$$ds^2 = R^2 d\phi^2 + dz^2$$

So,

$$ds = \sqrt{R^2\left(\frac{d\phi}{dz}\right)^2 + 1}\,dz$$

and

$$L = \int_{z_1}^{z_2} \underbrace{\sqrt{R^2(\phi')^2+1}}_{f(z,\phi,\phi') = f(\phi')}\,dz$$

E-L eqn:

$$\cancel{\frac{\partial f}{\partial \phi}}^{\,0} - \frac{d}{dz}\left(\frac{\partial f}{\partial \phi'}\right) = -\frac{d}{dz}\left(\frac{\partial f}{\partial \phi'}\right) = 0$$

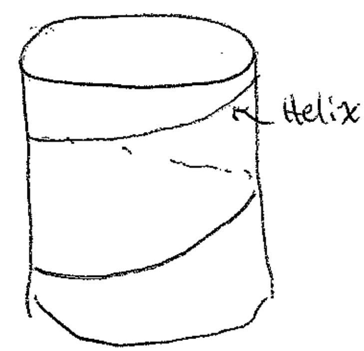
$$\Rightarrow \frac{\partial f}{\partial \phi'} = \text{const.} \equiv C_1 \Rightarrow \frac{1}{2}\frac{1}{\sqrt{R^2\phi'^2+1}}\cdot 2R^2\phi' = C_1$$

$$\text{or} \quad R^4\phi'^2 = C_1^2(R^2\phi'^2+1)$$

$$\Rightarrow (R^4 - C_1^2R^2)\,\phi'^2 = C_1^2$$

$$\Rightarrow \phi' = \text{const} \equiv \overset{\text{call it}}{C}$$

Then $\phi(z) = Cz + k$ ← constant Helix eqn.



$$\left.\begin{aligned}\phi(z_1) &= Cz_1 + k\\ \phi(z_2) &= Cz_2 + k\end{aligned}\right\}$$

Solve C and k.