Guest lectures on Math Methods

Day 2

Today

I. Logan's Guest Lecture:
Vector Integrals

II Yan's Guest Lecture:
Vector Integration by Parts & the Gradient in Polar Coords

III Cole's Guest Lecture:
Vector Derivatives in Curvilinear Coords

I Vector integrals

e.g.

line: $\int \vec{v} \cdot d\vec{l}$ $\quad d\vec{l} = dx\,\hat{l}$

surface: $\int \vec{v} \cdot d\vec{a}$ $\quad d\vec{a} = dx\,dy\,\hat{a}$

volume: $\int v\, d\tau$ $\quad d\tau = dx\,dy\,dz$

Closed versions:

$\oint \vec{v} \cdot d\vec{l}$

or

$\oint \vec{v} \cdot d\vec{a}$

Fundamental Theorem for Gradients:

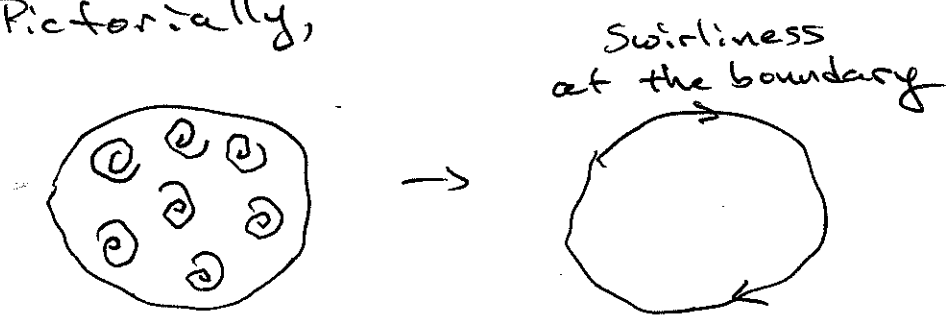
$$\int_a^b (\vec{\nabla} T) \cdot d\vec{l} = T(b) - T(a).$$

For curls (Stoke's Thm):

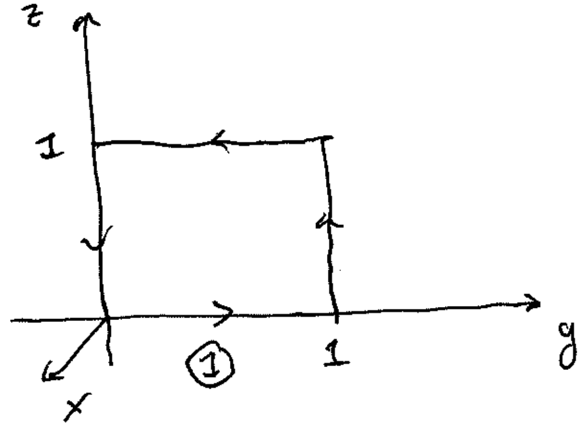
$$\int_a^b (\vec{\nabla} \times \vec{v}) \cdot d\vec{a} = \oint \vec{v} \cdot d\vec{l}$$

with $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$.

Pictorially,



Ex:



$\vec{v} = (2xz + 3y^2)\hat{y} + 4yz^2\,\hat{z}$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 2xz+3y^2 & 4yz^2 \end{vmatrix} = (4z^2 - 2x)\hat{x} + 2z\,\hat{z}$$

$$\int(\vec{\nabla}\times\vec{v})\cdot d\vec{a} = \int(4z^2 - \cancel{2x}^{\,0})\,dy\,dz$$

$$= \int_0^1\!\!\int_0^1 4z^2\,dy\,dz = \tfrac{4}{3}z^3\Big|_0^1 = \boxed{4/3}$$

Along 1 $z=0,\ x=0,\ dz=dx=0,$

$$\int\vec{v}\cdot d\vec{l} = \int_0^1(2x\cancel{z}^{\,0}+3y^2)\,dy$$

$$= \int_0^1 3y^2\,dy = y^3\Big|_0^1 = \boxed{1}$$

The answers for the rest are:

2 $\int_0^1\vec{v}\cdot d\vec{l} = \int_0^1 4z^2\,dz = \boxed{4/3}$

3 $\int_1^0\vec{v}\cdot d\vec{l} = -1$

4 $\int_1^0\vec{v}\cdot d\vec{l} = 0$

Then

1 + 2 + 3 + 4 $= \boxed{4/3}$

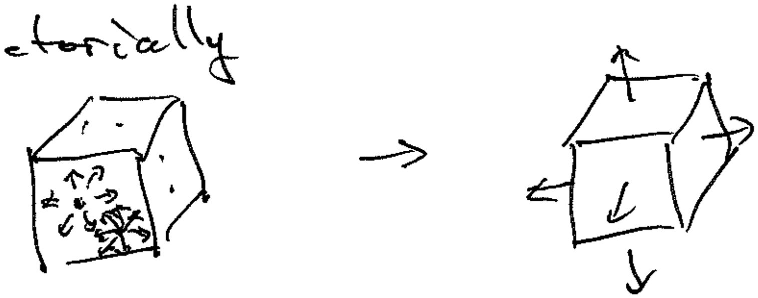
which agrees with previous result ✓

F.T. for Divergences

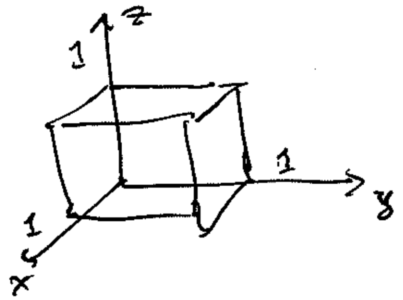
(Green's Thm, Div. Thm., Gauss' Thm.)

$$\int\vec{\nabla}\cdot\vec{v}\,d\tau = \oint\vec{v}\cdot d\vec{a}$$

Pictorially



Ex:



$$\vec{v} = y^2\,\hat{x} + (2xy + z^2)\,\hat{y} + 2yz\,\hat{z}$$

So,

$$\vec{\nabla}\cdot\vec{v} = 2x + 2y$$

and

$$\int(\vec{\nabla}\cdot\vec{v})\,d\tau = \int_0^1\int_0^1\int_0^1 (2x+2y)\,dx\,dy\,dz$$

and

$$\int \vec{v}\cdot d\vec{a} = \iint(-2x\cancel{y}^{\,0} - z^2)\,dx\,dz$$

$$= -\tfrac{1}{3}$$

Sum over all faces is 2, as it should be.

$$= \iint(1+2y)\,dy\,dz$$

$$= \int(2\,dz) = \boxed{2}$$

On front face,

$$x = 1,\quad dx = 0,\quad d\vec{a} = dy\,dz\,\hat{x}$$

$$\int \vec{v}\cdot d\vec{a} = \iint y^2\,dy\,dz = \tfrac{1}{3}$$

On left face

$$y = 0,\quad dy = 0 \quad d\vec{a} = dx\,dz\,(-\hat{y})$$

II Integration by parts

$$\int_a^b \frac{d}{dx}(fg)\,dx = (fg)\Big|_a^b$$

Product rule

$$= \int_a^b f\,\frac{dg}{dx}\,dx + \int_a^b \frac{df}{dx}\,g\,dx$$

Re-arranging,

$$\int_a^b f\,\frac{dg}{dx}\,dx = -\int_a^b \frac{df}{dx}\,g\,dx + (fg)\Big|_a^b$$

Combine int. by parts with fundamental Thms.

Div. Thm: $\int_V (\vec{\nabla}\cdot\vec{v})\, d\tau = \oint_S \vec{v}\cdot d\vec{a}$

So,

$$\int \vec{\nabla}\cdot(f\vec{A})\, d\tau = \oint f\vec{A}\cdot d\vec{a}$$

Note product rule

$$\vec{\nabla}\cdot(f\vec{A}) = \frac{\partial}{\partial x}(fA_x) + \frac{\partial}{\partial y}(fA_y) + \frac{\partial}{\partial z}(fA_z)$$

Then,

$$\int f(\vec{\nabla}\cdot\vec{A})\, d\tau + \int \vec{A}\cdot(\vec{\nabla}f)\, d\tau = \oint f\vec{A}\cdot d\vec{a}$$

$$\Rightarrow \boxed{\int f(\vec{\nabla}\cdot\vec{A})\, d\tau = -\int \vec{A}\cdot(\vec{\nabla}f)\, d\tau + \oint f\vec{A}\cdot d\vec{a}}$$

Another example would be

$$\oint_S f(\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \int_S [\vec{A}\times(\vec{\nabla}f)]\cdot d\vec{a} + \oint_P f\vec{A}\cdot d\vec{l}$$

Using normal product rule of functions

$$= \left[\underline{\frac{\partial f}{\partial x}A_x} + \frac{\partial A_x}{\partial x}f\right] + \left[\underline{\frac{\partial f}{\partial y}A_y} + f\frac{\partial A_y}{\partial y}\right] + \left[\underline{\frac{\partial f}{\partial z}A_z} + f\frac{\partial A_z}{\partial z}\right]$$

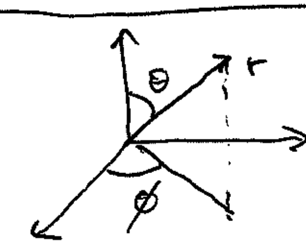
Collecting — and • terms

$$\Rightarrow \vec{\nabla}\cdot(f\vec{A}) = \vec{\nabla}f\cdot\vec{A} + f(\vec{\nabla}\cdot\vec{A})$$

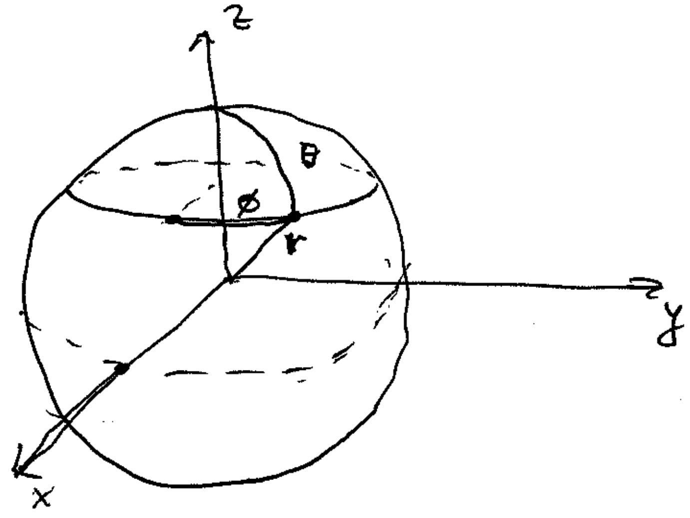
which you can derive using the curl theorem and the product rule

$$\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla}f).$$

Spherical Coord.s



Or another way of seeing it



In Cartesians:

$$\vec{\nabla} T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y}$$

In polars

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial T}{\partial \theta}\frac{\partial \theta}{\partial x}$$

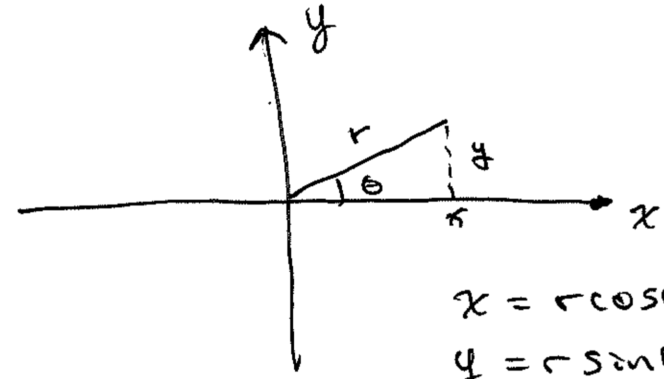
where

$$r = \sqrt{x^2+y^2}$$

and

$$\tan\theta = y/x$$

It's hard to find the differential operators in these coord.s. Let's try easier polar case



$$x = r\cos\theta$$
$$y = r\sin\theta$$

So,

$$\frac{\partial r}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2} \cdot 2x = \frac{x}{\sqrt{x^2+y^2}}$$
$$= \frac{r\cos\theta}{r} = \cos\theta$$

and

$$\frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$$ by similar calculation.

Then

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial r}\cos\theta - \frac{\partial T}{\partial \theta}\frac{\sin\theta}{r}$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial r}\sin\theta + \frac{\partial T}{\partial \theta}\frac{\cos\theta}{r}$$

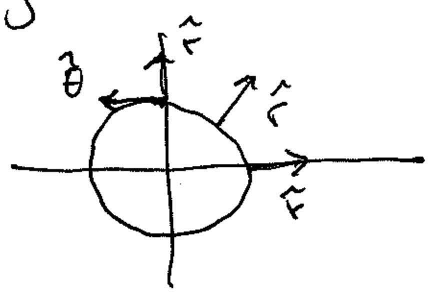
From the geometry

$$\hat{x} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$
$$\hat{y} = \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}$$

Final result

$$\vec{\nabla} T(r,\theta) = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta}$$

from pt. to pt., but still orthogonal



In general then

$$d\vec{l} = f\,du\,\hat{u} + g\,dv\,\hat{v} + h\,dw\,\hat{w}$$

The coeff.s f, g, and h are

III Curvilinear coord.s

Let's call our coordinates

u, v, and w.

We'll require the basis to be orthogonal

$$\hat{u}\cdot\hat{v} = 0 \quad \text{and similar.}$$

E.g. In polar coord.s the basis vectors are varying

Scale factors and are in general functions of $\{u, v, w\}$.

For the case of spherical coord.s this is

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\,S\theta\,d\phi\,\hat{\phi}$$

Differentials: Given $t(u,v,w)$

$$dt = \frac{\partial t}{\partial u}du + \frac{\partial t}{\partial v}dv + \frac{\partial t}{\partial w}dw$$
$$= d\vec{l}\cdot\vec{\nabla}t = |d\vec{l}|\,|\vec{\nabla}t|\cos\theta$$

In general let's call

$$\vec{\nabla} t = \alpha \hat{u} + \beta \hat{v} + \gamma \hat{w}.$$

Then,

$$dt = \vec{dl} \cdot \vec{\nabla} t$$

$$= f \cdot \alpha\, du + g\beta\, dv + h\gamma\, dw$$

But, the two dt expressions should agree, and so

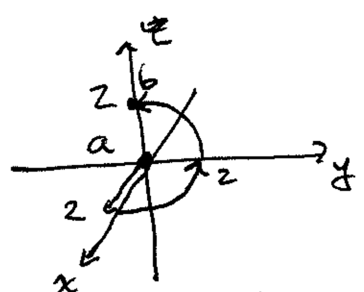
$$\vec{\nabla} t = \frac{1}{f}\frac{\partial t}{\partial u}\hat{u} + \frac{1}{g}\frac{\partial t}{\partial v}\hat{v} + \frac{1}{h}\frac{\partial t}{\partial w}\hat{w}.$$

Consider the gradient thm.

$$\int_a^b \vec{\nabla} t \cdot \vec{dl} = t(b) - t(a).$$

Let's do an example in Sph. Coords

See fig. below.



$$a = \begin{matrix}(r, & \theta, & \phi)\\ (0, & \frac{\pi}{2}, & 0)\end{matrix}$$

$$b = (2, 0, \tfrac{\pi}{2})$$

$$t(r,\theta,\phi) = r[c\theta + s\theta c\phi]$$

$$t(b) - t(a) = 2 - 0 = 2$$

From the scale factor result

$$\vec{\nabla} t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{rs\theta}\frac{\partial t}{\partial \phi}\hat{\phi}$$

$$= [c\theta + s\theta c\phi]\hat{r} + [c\phi s\theta - s\theta]\hat{\theta} + [-s\phi]\hat{\phi}$$

Then

$$\int_{\gamma_1} \vec{\nabla} t \cdot \vec{dl} = \int_0^2 dr = r\Big|_0^2 = \boxed{2.}$$

Full theorem works out.