E & M

Day 3

Today

I Logistics

II 15 mins. on Young's Ch. "Retrieval"

III Nathalie's Guest Lecture: 3D delta-functions

IV Andrew's Guest Lecture: Theory of Vector Fields & Helmholtz Theorem

Satisfies

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

or also

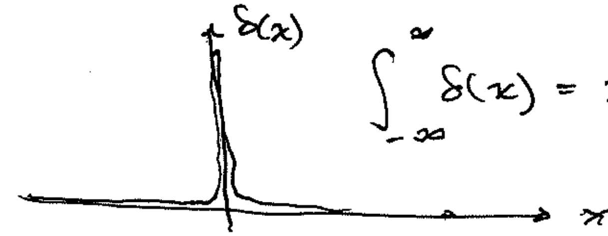
$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

E.g.

$$\int_{-\infty}^{\infty} (x^2+2)\,\delta(x)\,dx = 2.$$

III 1D delta-function

- Not a function, but a distribution



$$\int_{-\infty}^{\infty} \delta(x) = 1$$

- Zero everywhere except $x=0$

$$\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty & x = 0 \end{cases}$$

3D delta-function

- Zero at all pts except $(0,0,0)$

$$\int_{\mathbb{R}^3} \delta^3(\vec{r})\,d\tau = 1$$

Generalizes 1D result

$$\int_{\mathbb{R}^3} f(\vec{r})\,\delta(\vec{r}-\vec{a})\,d\tau = f(\vec{a})$$

Note that the notation is

Condensed

$$\delta(\vec{r}-\vec{r}\,') = \delta(x-x')\delta(y-y')\delta(z-z')$$

Let's try to find the Fourier transform of the 3D δ:

1D: $$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \delta(x-x')e^{-ikx}\,dx$$

3D: $$F(\vec{k}) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int_{-\infty}^{\infty} \delta(\vec{r}-\vec{r}\,')e^{-i\vec{k}\cdot\vec{r}}\,d\tau$$

Simplifying

$$F(\vec{k}) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 e^{-i\vec{k}\cdot\vec{r}\,'}$$

Inverse Fourier transforming:

In 1D

$$f(x) = \delta(x-x') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk$$

From above

$$F(k) = \frac{1}{\sqrt{2\pi}}e^{-ikx'},$$

Note that

$$\vec{k}\cdot\vec{r} = k_x x + k_y y + k_z z$$

so,

$$e^{-i\vec{k}\cdot\vec{r}} = e^{-ik_x x}e^{-ik_y y}e^{-ik_z z}$$

Then

$$F(\vec{k}) = \left(\frac{1}{\sqrt{2\pi}}\right)^3 \iiint_{-\infty}^{\infty} \delta(x-x')\delta(y-y')\delta(z-z')\, e^{-ik_x x}e^{-ik_y y}e^{-ik_z z}\,dx\,dy\,dz$$

$$= \left(\frac{1}{\sqrt{2\pi}}\right) e^{-ik_x x'}e^{-ik_y y'}e^{-ik_z z'}$$

so,

$$\delta(x-x') = \left(\frac{1}{\sqrt{2\pi}}\right)^2 \int_{-\infty}^{\infty} e^{-ikx'}e^{ikx}\,dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x')}\,dk$$

or in 3D

$$\delta(\vec{r}-\vec{r}\,') = \iiint_{-\infty}^{\infty} e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}\,d^3k = \iiint_{-\infty}^{\infty} e^{i\vec{k}\cdot\vec{\mathscr{r}}}\,d^3k$$

On the HW you showed

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 0$$

and also

$$\int \vec{\nabla} \cdot \frac{\hat{r}}{r^2}\, d\tau = 4\pi$$

How can these be consistent?

Go to Green's Thm

$$\int \vec{\nabla} \cdot \frac{\hat{r}}{r^2}\, d\tau = \oint \frac{\hat{r}}{r^2} \cdot d\vec{a}$$

of vector derivatives, e.g.

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \text{(no magnetic monopoles)}$$

We want to understand what these kinds of eqn.s tell us.

There are two main theorems:

i) Curl-less field

a) $\vec{\nabla} \times \vec{F} = \vec{0}$ everywhere

b) $\int_a^b \vec{F} \cdot d\vec{l}$ is independent of path for any end pts.

where

$$d\vec{a} = r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}$$

So,

$$\oint_{S^2} \frac{\hat{r}}{r^2} \cdot \left(r^2 \sin\theta\, d\theta\, d\phi\, \hat{r}\right) = 4\pi$$

We should have said

$$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 0 \quad \text{for all } r \neq 0$$

IV Maxwell's eqn.s provide a nice example of physical differential equations expressed in terms

c) $\oint \vec{F} \cdot d\vec{l} = 0$ for any closed loop

d) $\vec{F}$ is the gradient of a scalar field

ii) Divergence-less fields

a) $\vec{\nabla} \cdot \vec{F} = 0$ everywhere

b) $\int \vec{F} \cdot d\vec{a}$ is indep. of surface for any bndry line

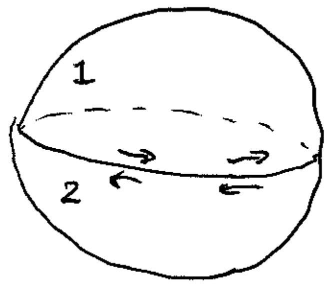
c) $\oint \vec{F} \cdot d\vec{a} = 0$ for any closed surface

d) $\vec{F}$ is the curl of some vector func. $\vec{A}$

$$\boxed{\vec{\nabla} \cdot \vec{F} = 0 \iff \vec{F} = \vec{\nabla} \times \vec{A}}$$

I'll prove ii) (c) $\Rightarrow$ (b)

Consider an example, the sphere. Cut it in half along the equator.



$$\oint_{S^2} \vec{F}\cdot d\vec{a} = \int_{\underset{\rightarrow}{1}} \vec{F}\cdot d\vec{a} + \oint_{\underset{\leftarrow}{2}} \vec{F}\cdot d\vec{a}$$

$= 0$ (by condition (c))

Note that the right-hand rule and outward orientation mean that at the boundary the integrals 1 and 2 are oppositely oriented, indicated by arrows

$$\Rightarrow \int_{\underset{\rightarrow}{1}} \vec{F}\cdot d\vec{a} = -\int_{\underset{\leftarrow}{2}} \vec{F}\cdot d\vec{a} = \int_{\underset{\rightarrow}{2}} \vec{F}\cdot d\vec{a}$$

But, the arbitrariness of the surface and of the boundary choice means that the integral is independent of the surface once you've fixed its boundary.

Helmholtz Theorem:

How can we describe $\vec{F}$ in terms of $D = \vec{\nabla}\cdot\vec{F}$ and $\vec{C} = \vec{\nabla}\times\vec{F}$?

It is uniquely determined given boundary conditions by

$$\vec{F} = -\vec{\nabla}U + \vec{\nabla}\times\vec{W}.$$

Here

$$\vec{W}(\vec{r}) = \frac{1}{4\pi}\int \frac{\vec{C}(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

and

$$U(\vec{r}) = \frac{1}{4\pi}\int \frac{D(\vec{r}\,')}{\mathscr{r}}\,d\tau'.$$

Let's unpack the notation for

$$\vec{W}: W_x(x,y,z) = \frac{1}{4\pi}\iiint \frac{C_x(x',y',z')}{\mathscr{r}}\,dx'dy'dz'$$

$$= \frac{1}{4\pi}\iiint \frac{C_x(x',y',z')\,dx'dy'dz'}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}$$

$W_y(x,y,z)$

$W_z(x,y,z)$