Thermal Physics

Nov. 24^{th}, 2014

I Last time

II Fermi-Dirac Statistics

- Found that

$$n_s = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \epsilon_s}$$

- Discovered ~~constraints~~ constraint on most cases

MB: $n_r = 0, 1, 2, \dots$

$\sum_r n_r = N$

BE: $n_r = 0, 1, 2, \dots \quad \sum_r n_r = N$

I. • Wave function symmetry

MB-stats no symmetry

BE-stats $\psi(\dots Q_j \dots Q_i \dots)$

$= \psi(\dots Q_i \dots Q_j)$ & integer spin

FD-stats half-integer spin

$$\psi(\dots Q_j \dots Q_i \dots) = -\psi(\dots Q_i \dots Q_j \dots)$$

Except photons: $n_r = 0, 1, 2, \dots$ but no constraint!

FD: $n_r = 0, 1 \quad \sum_r n_r = N$

- Reformulated

$$\bar{n}_s = \frac{\sum_R n_s e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \dots)}}{\sum_R e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \dots)}}$$

$$= \frac{\sum_{n_1, n_2, \dots} n_s e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \dots)}}{\sum_{n_1, n_2, \dots} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \dots)}}$$

Then

$$\bar{n}_s = \frac{\sum_{n_s} n_s e^{-\beta n_s \epsilon_s} \sum^{(s)}_{n_1, n_2, \ldots} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}}{\sum_{n_s} e^{-\beta n_s \epsilon_s} \sum^{(s)}_{n_1, n_2, \ldots} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}}$$

These factors don't necessarily cancel — easiest to see in examples to come, so let's do 'em.

Simplest is photon because there is <u>no</u> constraint and sums in numerator and denominator cancel.

$$\bar{n}_s = \frac{\sum_{n_s} n_s e^{-\beta n_s \epsilon_s}}{\sum_{n_s} e^{-\beta n_s \epsilon_s}}$$

so

$$\bar{n}_s = -\frac{1}{\beta}\frac{\partial}{\partial \epsilon_s} \ln\left(\sum e^{-\beta n_s \epsilon_s}\right)$$

$$= -\frac{1}{\beta}\frac{\partial}{\partial \epsilon_s} \ln\left(\left[1 - e^{-\beta\epsilon_s}\right]^{-1}\right)$$

$$= \frac{e^{-\beta\epsilon_s}}{1 - e^{-\beta\epsilon_s}}$$

or

$$\boxed{\bar{n}_s = \frac{1}{e^{\beta\epsilon_s} - 1}}$$

Planck distribution.

II FD statistics has the constraint

$$\sum_r n_r = N$$

If state s has particle then $\sum^{(s)}$ is over $(N-1)$ particles.

Useful abbreviation

$$Z_s(N) = \sum^{(s)}_{n_1, n_2, \ldots} e^{-\beta(n_1\epsilon_1 + n_2\epsilon_2 + \cdots)}$$

all N particles in sum.

For FD then $n_s = 0$ or 1 and

$$\bar{n}_s = \frac{0 \cdot Z_s(N) + e^{-\beta \epsilon_s} Z_s(N-1)}{Z_s(N) + e^{-\beta \epsilon_s} Z_s(N-1)}$$

$$= \frac{1}{\left[\frac{Z_s(N)}{Z_s(N-1)}\right] e^{\beta \epsilon_s} + 1}$$

How does Z_s depend on N? Well, Z_s and the result shouldn't depend strongly on which is omitted, so

$$\alpha_s = \alpha = \frac{\partial \ln Z}{\partial N}$$

Now taking $\Delta N = 1$ we have

$$Z_s(N-1) = Z_s(N)\, e^{-\alpha}$$

$$\ln Z_s(N - \Delta N) \simeq \ln Z_s(N) - \frac{\partial \ln Z_s}{\partial N} \Delta N$$

$$\equiv \ln Z_s(N) - \alpha_s \Delta N$$

with

$$\alpha_s \equiv \frac{\partial \ln Z_s}{\partial N}$$

Then

$$Z_s(N - \Delta N) = Z_s(N)\, e^{-\alpha \Delta N}$$

But, there are many, many states ~~Z_s~~ contributing to and

$$\bar{n}_s = \frac{1}{e^{\alpha + \beta \epsilon_s} + 1}$$

Two aspects of α:

Finding α in practice:

Use the constraint

$$\sum_r \bar{n}_r = N$$

or

$$\sum_r \frac{1}{e^{\alpha + \beta \epsilon_r} + 1} = N$$

Interpreting α: Well

$$\alpha = \frac{\partial \ln Z}{\partial N}$$

$$= \frac{\partial}{\partial N}\left(\frac{1}{-kT} F\right)$$

$$= -\frac{1}{kT}\frac{\partial F}{\partial N}$$

$$= -\beta\mu$$

Recall that the validity of the semiclassical approximation relied on

$$\left(\frac{V}{N}\right)^{1/3} \gg \lambda_{th} = \frac{h}{\sqrt{3mkT}}$$

For electrons in a metal

$$\bar{\lambda} \approx 50 \times 10^{-8}\ \text{cm}$$

and the average separation is

$$\bar{R} \approx 2 \times 10^{-8}\ \text{cm}$$

and the semiclassical approximation breaks down — we have to treat them as fermions.

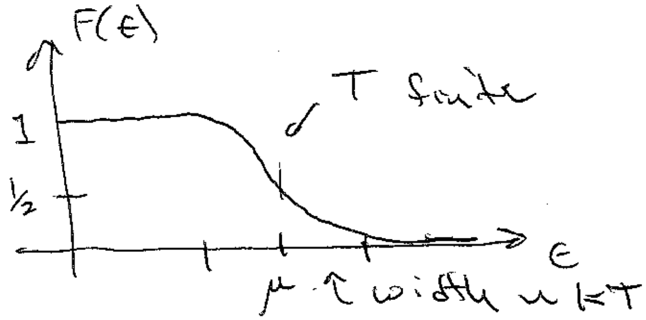
So,

$$\boxed{\bar{n}_s = \frac{1}{e^{\beta(\epsilon_s - \mu)} + 1}}$$

III Skip to Section 9.16 — Conduction electrons in a metal

Let's look at the "Fermi function"

If $\frac{\mu}{kT} \gg 1$ then if $\epsilon \ll \mu$ we have $\beta(\epsilon-\mu) \ll 0$ and $F(\epsilon) = 1$, while for $\epsilon \gg \mu$ then $\beta(\epsilon-\mu) \gg 0$ and $F(\epsilon) \sim e^{\beta(\mu-\epsilon)}$ an exponential decay



At $T=0$ this becomes extreme and