I Last time

II Bose-Einstein Condensation

- Found effective contribution of electrons to heat capacity

$$C_v \approx \frac{3}{2} N k \frac{kT}{\mu}$$

$$= \frac{3}{2} N k \frac{T}{T_F}, \quad T_F \equiv \frac{\mu}{k}$$

At room temp. electrons don't contribute (e.g. for Copper $T_F \sim 80{,}000^\circ K$)

$\Rightarrow$ heat capacity due to phonons.

- Counted states to find

$$\rho\, d^3\vec{k} = \frac{V}{(2\pi)^3} d^3\vec{k}$$

and filled them to get

$$k_F = \left(3\pi^2 \frac{N}{V}\right)^{1/3}$$

which led to

$$\mu_0 = \frac{\hbar^2}{2m} k_F^2 = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}$$

- Derived Bose-Einstein distribution

$$\boxed{\bar{n}_s = \frac{1}{e^{\beta(\epsilon_s - \mu)} - 1}}$$

Bose-Einstein Statistics.

Can remember sign through # of particles in state requirement.

II Finding μ is trickiest in this case. But, the outcome is very interesting — as you cool below a critical temperature, the bosons suddenly condense into the ground state.

First consider $T \to 0$, all the atoms will accumulate in the ground state,

$$\bar{n}_0 = \frac{1}{e^{(\epsilon_0 - \mu)/kT} - 1}$$

For low T, $\bar{n}_0$ large, so

$$e^{(\epsilon_0-\mu)/kT} \approx 1$$

or its exponent is small. So we can Taylor expand to get

$$\bar{n}_0 \approx \frac{1}{1 + \frac{\epsilon_0 - \mu}{kT} - 1} = \frac{kT}{\epsilon_0 - \mu}$$

As an estimate of the energy take a standing matter wave in a 3D box

$$\epsilon_0 = \frac{p^2}{2m} = \frac{h^2}{2m\lambda_x^2} + \frac{h^2}{2m\lambda_y^2} + \frac{h^2}{2m\lambda_z^2} \quad \lambda = 2L$$

$$= \frac{3h^2}{8mL^2}$$

Then we have

At $\underline{T=0}$, we get

$$\epsilon_0 = \mu$$

and when $T \neq 0$ but small μ must be just a little less than ϵ_0. So, at what temp. does this begin to happen? Recall, we fix μ using

$$N = \sum_s \frac{1}{e^{\beta(\epsilon_s - \mu)} - 1}$$

Not a sum that I know a good technique for, so approximate.

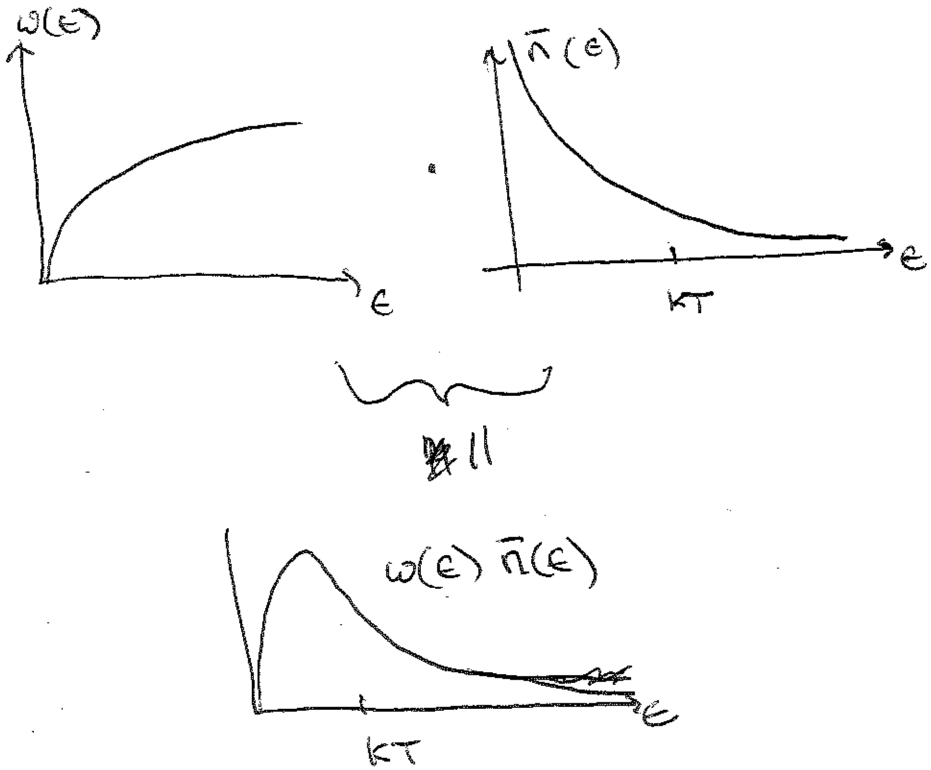
Use

$$N = \int_0^\infty \underbrace{\frac{1}{e^{\beta(\epsilon-\mu)}-1}}_{\text{particles in these states}} \underbrace{\omega(\epsilon)}_{\text{density of state}} d\epsilon$$

$\omega(\epsilon)\,d\epsilon$: # of states

We want density of states in energy now — how do we get it?

$$\#\text{ states} = \frac{V \cdot d^3p}{h^3} = \frac{V}{h^3} 4\pi p^2 dp$$



But

$$\epsilon = \frac{p^2}{2m}$$

and so

$$d\epsilon = \frac{2p}{2m} dp$$

And

$$\#\text{ states} = \frac{V}{h^3} 4\pi \sqrt{2m\epsilon}\, m\, d\epsilon$$

$$= \underbrace{\frac{4\pi m^{3/2}}{h^3} V \sqrt{2\epsilon}}_{\omega(\epsilon)}\, d\epsilon$$

Unfortunately, we still can't do the integral

$$N = \int_0^\infty \frac{1}{e^{\beta(\epsilon-\mu)}-1} \frac{4\pi m^{3/2}\sqrt{2}}{h^3} V \sqrt{\epsilon}\, d\epsilon$$

We guess μ until we get right value. What about $\mu = 0$?

$$N = \frac{4\pi m^{3/2}\sqrt{2}V}{h^3} \int_0^\infty \frac{\sqrt{\epsilon}\, d\epsilon}{e^{\epsilon/kT}-1}$$

Then $x = \epsilon/kT \quad dx = \frac{d\epsilon}{kT}$

$$N = \frac{4\pi m^{3/2}\sqrt{2}\,V}{h^3}(kT)^{3/2}\underbrace{\int_0^\infty \frac{\sqrt{x}\,dx}{e^x - 1}}_{\# = 2.315}$$

So,

$$N = 2.315\left(\frac{4\pi m^{3/2}\sqrt{2}\,V}{h^3}(kT)^{3/2}\right)$$

Clearly wrong since N is not T dependent.

Only holds at one temperature, call it T_c.

$$kT_c = 0.527\left(\frac{h^2}{2\pi m}\right)\left(\frac{N}{V}\right)^{2/3}$$

Above T_c the integral better get smaller so that N from here stays fixed.

This means $\mu < 0$, to get smaller integral.

Below T_c we have to be more careful; Replacing the sum by the integral is no good. The trouble is that the integral approximation doesn't do a good job with the ground state, which is now highly populated!

(To see this note that the density of states $\propto \sqrt{\epsilon} \xrightarrow{\epsilon \to 0} 0$ kills contributions at low energy.

It's not a good approximation to the discrete spacing here)

However, this integral should still capture most of the particles in <u>excited</u> states

$$N_{excite} = 2.612\left(\frac{2\pi m kT}{h^2}\right)^{3/2} V$$

for $T < T_c$

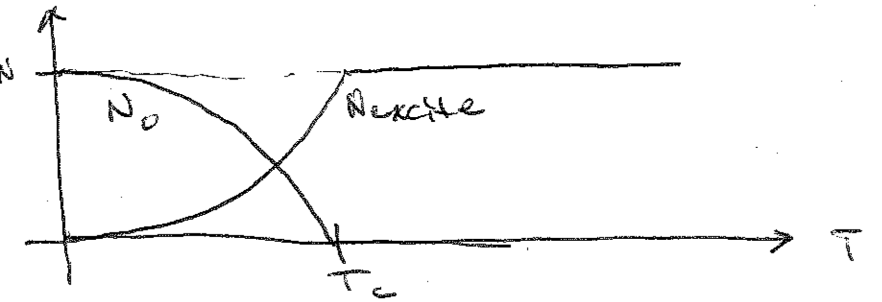
More simply

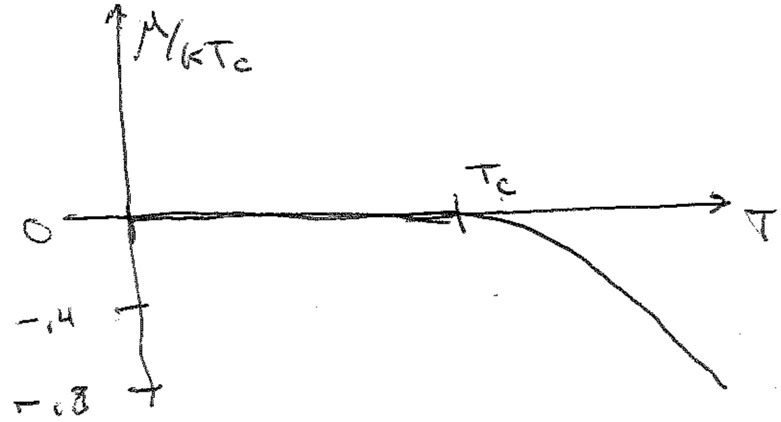
$$N_{excite} = \left(\frac{T}{T_c}\right)^{3/2} N \quad (T < T_c)$$

and

$$N_0 = N - N_{excite} = \left[1 - \left(\frac{T}{T_c}\right)^{3/2}\right] N$$

$$(T < T_c)$$





$$\lambda_{Th}^3 = \left(\frac{h^2}{2\pi m k T}\right)^{3/2}$$