Thermal Physics

~~Lecture~~ Meeting III

I Last time

II Many variables & functions of random variables

III Review of some salient features of classical & quantum mechanics

IV Statistics and physics?

I. Last time

- Binomial Distribution

$$W_N(n) = \frac{N!}{n!(N-n)!} p^n q^{N-n}$$

- Average $\bar{u} = \sum_i u_i P(u_i)$

Dispersion $\overline{\Delta u^2} = \overline{(u-\bar{u})^2}$
$= \overline{u^2} - \bar{u}^2$

- "Probability density" is

integrated to find probabilities. Has inverse units to the random variable under consideration

$$\text{Prob} = \int_{x_i}^{x_f} P(x)\,dx,$$

II. Consider

$u_i \quad i = 1, \ldots, M$

and $v_j \quad j = 1, \ldots, N$

$P(u_i, v_j)$ probability of u_i and v_j

Prob. of $u = u_i$ irrespective of v_j

$$P_u(u_i) = \sum_j P(u_i, v_j).$$

If u and v are statistically independent

$$P(u_i, v_j) = P_u(u_i) P_v(v_j)$$

In this (special) case:

$$\overline{f(u)g(v)} = \sum_{ij} P(u_i, v_j) f(u_i) g(v_j)$$

$$= \sum_{ij} P_u(u_i) P_v(v_j) f(u_i) g(v_j) = \overline{f(u)}\,\overline{g(v)}$$

Functions of random variables:

Give $\varphi[u]$ and $P(u)\,du$ what is the probability of φ being in the range φ to $\varphi + d\varphi$?

$$W(\varphi)\,d\varphi = \int_{d\varphi} P(u)\,du$$

This means "integrate over u s.t. $\varphi \in [\varphi, \varphi + d\varphi]$". So,

$$W(\varphi)\,d\varphi = \int_{\varphi}^{\varphi+d\varphi} P(u)\left|\frac{du}{d\varphi}\right| d\varphi$$

$$= P(u)\left|\frac{du}{d\varphi}\right| d\varphi$$

Common example: Frequency as a function of wavelength

$$\nu(\lambda) = \frac{c}{\lambda}$$

You often want to go back and forth when thinking about black body spectra, so

$$W(\lambda)\,d\lambda = P(\nu)\,\frac{d\nu}{d\lambda}\,d\lambda = P(\nu(\lambda))\left(-\frac{c}{\lambda^2}\right)d\lambda.$$

III Classical Mechanics

Newton: $a = \dfrac{d^2x}{dt^2} = \dfrac{F(x)}{m}$ — 2^{nd} order ODE, Cartesian coords. Initial data $(x(0), \dot{x}(0))$

Lagrange: $\dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}}\right) = \dfrac{\partial L}{\partial q}$ — 2^{nd} order ODE, general coord.s. Initial data $(q(0), \dot{q}(0))$

$$L = T - V = L(q, \dot{q})$$

Kinetic energy ↗ ↖ potential energy

Huge advantages: general coords are very adaptable, a single scalar determines all the equations of motion (EOM).

Hamilton: gen. momentum $p = \dfrac{\partial L}{\partial \dot{q}}$

$$H(q,p) = p\dot{q} - L$$

$$= T + V$$

↖ whenever (roughly) coords and L are t-independent

$$\dot{q} = \frac{\partial H}{\partial p}\ ; \quad \dot{p} = -\frac{\partial H}{\partial q}$$

1^{st} order ODE, general coords

q and p are independent! Initial data $q(0), p(0)$

No constraint like $\cancel{\dot{q}} = p$.

Call the space of (q_i, p_i) $i = 1, \cdots, f$ the phase space. Uniqueness of solutions of 1st order ODEs implies that trajectories in phase space never cross.

Example: Oscillator with $m = 1$,

$$T = \frac{p^2}{2m} = \frac{1}{2}p^2 \qquad U = \frac{1}{2}m\omega^2 q^2 = \frac{1}{2}q^2\omega$$

$$H = \frac{1}{2}(\omega q^2 + p^2)$$

These different descriptions are surprisingly useful; for us, the last one is particularly so. Why?

Quantum Mechanics

Bohr's remarkable discovery,

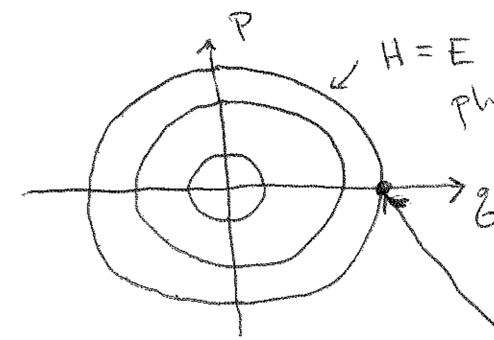
boundary conditions

force ang. momentum

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p}$$

$$= -\dot{p}\dot{q} + \dot{q}\dot{p} = 0$$

So, $H = \overset{\text{value } E}{E}$ is conserved.



H = E is the phase space picture of the trajectory.

Starting here

to have discrete set of values

$$L = n\hbar \qquad n = 1, 2, \cdots$$

Physical input 1: Observables (like L) that appeared to be continuous classically can be discrete, taking only certain values, in quantum mechanics.

Which observables? A hunt began for which observables and why.

In a 2D phase space the result is easy to state. Consider ~~an~~ classical observable $A(q,p)$ (e.g. energy), ~~and~~ if the level set $A = a$ ← some number captures a finite area in phase space then the observable A is quantized.

The area inside $A=a$ is

$$\text{Area} = \int_{\Omega} dq\, dp$$

Stoke's theorem ↓

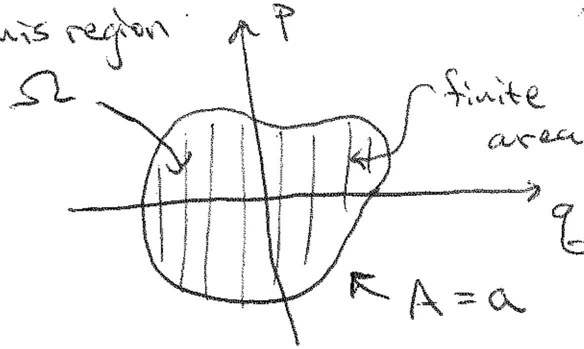
$$= \oint_{\partial\Omega} p\, dq$$

$\partial\Omega$ = curve where $A=a$

The approximation they found was

$$S(A) \equiv \oint_{\partial\Omega} p\, dq = (n+\tfrac{1}{2})h \quad \leftarrow \text{Planck's constant}$$
$$= 2\pi(n+\tfrac{1}{2})\hbar$$

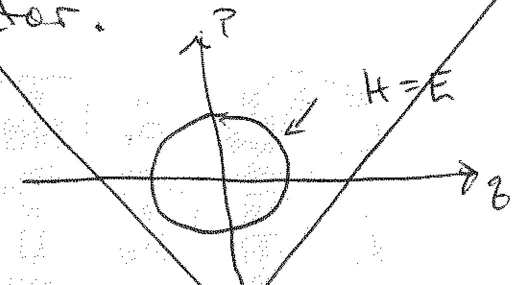
call this region Ω



~~with quantized values~~

They even discovered an approximation for the quantized values.

Let's try it on our harmonic oscillator.



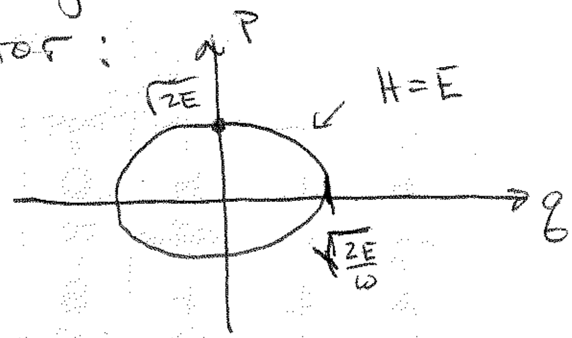
So $E = \frac{1}{2}(\omega^2 q^2 + p^2)$

$\Rightarrow$ circle of radius $\sqrt{2E}$

Then area $= \pi(\text{radius})^2$
$= 2\pi E$

So,
$2\pi E = 2\pi(n+\tfrac{1}{2})\hbar$

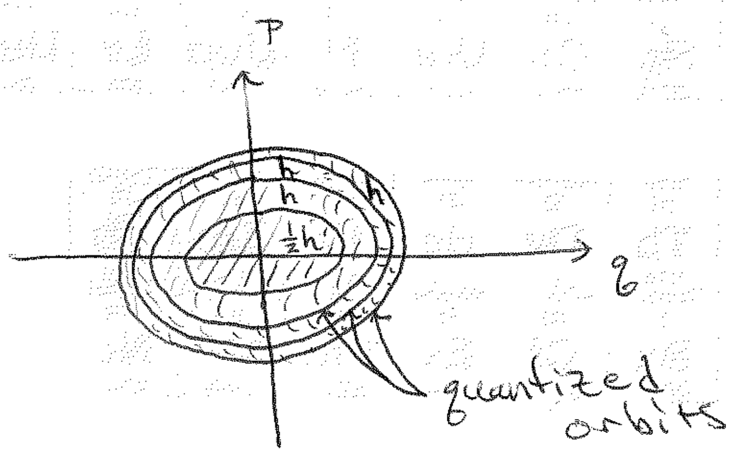
Let's try it on our harmonic oscillator:



So $E = \frac{1}{2}(\omega^2 q^2 + p^2)$

$\Rightarrow$ area of ellipse $= \pi\sqrt{2E} \cdot \sqrt{\frac{2E}{\omega}}$

$= \pi \frac{2E}{\omega}$



<u>Physical input 2</u>: Measurement of a system inevitably disturbs its state; so, you can't know the position and momentum of a physical

So,

$$\frac{2\pi E}{\omega} = 2\pi(n+\tfrac{1}{2})\hbar$$

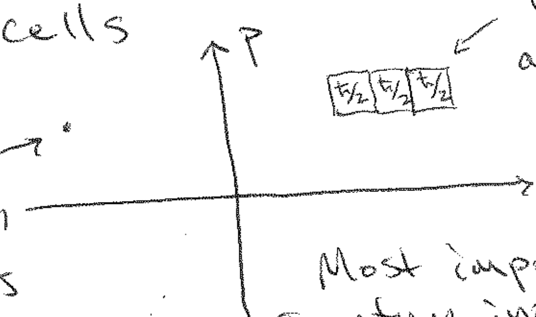
or $\boxed{E = (n+\tfrac{1}{2})\hbar\omega}$ Exact result!

The energy of the harmonic osc. is quantized! In general gives an excellent approximation that gets better and better the larger n is.

System to arbitrary precision:

$$\Delta q \, \Delta p \geq \frac{\hbar}{2}$$

Can design states that saturate this bound (called coherent states). They divide phase space into cells



Instead of a continuum of points

We get a discrete set of cells.

Most important quantum input for us!

<u>Physical input 3</u>: Nature is intrinsically probabilistic. Describe quantum states by smooth functions $\Psi(q)$ and expected outcomes of measurements, say of $\hat{A}$, by

$$\langle \hat{A} \rangle = \int_{-\infty}^{\infty} \Psi^*(q) \hat{A}\, \Psi(q)\, dq.$$

(Don't need to know this, its here for completeness.)

Trial	Outcome
#1	H H T
#2	T T T
#3	H T H
#4	H T T
#5	T H H

We call the detailed specification of the particular state the "microstate" — it includes all micro details.

IV How is all of this useful to us?

A little terminology helps in answering this question.

Consider three coin flips with heads = H, tails = T

Now run several trials

In contrast say we had an apparatus that was only sensitive to the total number of heads and tails, e.g.

2H and 1T

This is called a macrostate.

Assuming fair coins

the probability of this macrostate is

$$P_3(2) = \frac{3!}{2!\,1!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1$$

$$= 3 \cdot \frac{1}{8} = \boxed{3/8}$$

The advantage of the quantum mechanical inputs 1 and 2 is that they allow us to delineate the microstates in detail.

Basic postulate

Isolated system's conserve energy. What is the relative probability of finding the system in a state consistent with this energy?