Homework 10
Due by 10pm on Wednesday, December 2nd, 2020

Reread any part of Schroeder's Ch. 6 that didn't make sense, and read Ch. 7, Secs. 7.1-7.3. Notes.

1. The methods you have learned around Boltzmann factors can even be used to estimate proper-ties of the very early universe, when it was at extremely high temperatures. During this epoch the proton and the neutron can be thought of as two different states of the same particle, called the "nucleon." (The reactions that convert a proton to a neutron or vice versa require the absorption of an electron or a positron or a neutrino, but all of these particles tend to be very abundant at sufficiently high temperatures.) Since the neutron's mass is higher than the proton's by 2.3×10^{30} kg, its energy is higher by this amount times c^2. Suppose, then, that at some very early time, the nucleons were in thermal equilibrium with the rest of the universe at 10^{11} K. What fraction of the nucleons at that time were protons, and what fraction were neutrons?

2. (a) Prove in general that
$$\sigma_E^2 = \overline{E^2} - \overline{E}^2,$$
that is, the standard deviation squared is the average of the squares minus the square of the aver-age. This formula often gives the easiest way to compute a standard deviation.

(b) Given a system in equilibrium with a reservoir at temperature T, the average value of E^2 is
$$\overline{E^2} = \frac{1}{Z}\frac{\partial^2 Z}{\partial \beta^2}.$$

Use this result and that of the previous part to derive a formula for σ_E in terms of the heat capacity, $C = \partial \overline{E}/\partial T$. You should find
$$\sigma_E = kT\sqrt{C/k}.$$

(c) Use (b) to obtain a formula for the standard deviation of the energy of a system of N identical harmonic oscillators (such as in an Einstein solid), in the high-temperature limit. Divide by the average energy to obtain a measure of the fractional fluctuation in energy. Evaluate this fraction numerically for $N = 1, 10^4$, and 10^{20}. Discuss the results.

3. (a) Use the Maxwell-Boltzmann distribution to calculate the average value of v^2 for the molecules in an ideal gas. Check that your answer agrees with what we found from equipartition and the ideal gas law.

(b) Verify from the Maxwell-Boltzmann speed distribution that the most likely speed of a molecule is $\sqrt{2kT/m}$.

4. Imagine space consisted of just two dimensions, but the laws of physics are otherwise the same. Derive the speed distribution formula for an ideal gas of nonrelativistic particles in this fictitious world, and sketch this distribution. Carefully explain the similarities and differences between the two-dimensional and three-dimensional cases. What is the most likely velocity vector? What is the most likely speed?

5. Claude Shannon and many researchers since have defined entropy by the alternative formula

$$S = -k \sum_s \mathcal{P}(s) \ln \mathcal{P}(s),$$

where the sum runs over all microstates accessible to the system and $\mathcal{P}(s)$ is the probability of the system being in microstate s.

(a) For an isolated system, $\mathcal{P}(s) = 1/\Omega$ for all accessible states s. Show that in this case the preceding formula reduces to our familiar definition of entropy.

(b) For a system in thermal equilibrium with a reservoir at temperature T, $\mathcal{P}(s) = e^{E(s)/kT}/Z$. Show that in this case as well, the preceding formula agrees with what we've recently learned about entropy.

6. Consider a large system of N indistinguishable, noninteracting molecules (perhaps in an ideal gas or a dilute solution). Find an expression for the Helmholtz free energy of this system, in terms of Z_1, the partition function for a single molecule. (Use Stirling's approximation to eliminate the $N!$.) Then use your result to find the chemical potential, again in terms of Z_1.

7. For a diatomic gas near room temperature, the internal partition function is simply the rotational partition function computed in Section 6.2 of Schroeder, multiplied by the degeneracy Z_e of the electronic ground state.

(a) Show that the entropy in this case is

$$S = Nk \left[\ln \left(\frac{V Z_e Z_{\text{rot}}}{N v_Q} \right) + \frac{7}{2} \right].$$

Calculate the entropy of a mole of oxygen ($Z_e = 3$) at room temperature and atmospheric pressure, and compare to the measured value in the table at the back of Schroder's book.

(b) Calculate the chemical potential of oxygen in earth's atmosphere near sea level, at room temperature. Express the answer in electron-volts.

(c) For a mole of nitrogen (N_2) gas at room temperature and atmospheric pressure, compute the following: U, H, F, G, S, and μ. (The electronic ground state of nitrogen is not degenerate.)