Homework 11

Due by 10pm on Wednesday, December 9th, 2020

Read Ch. 7. Notes.

1. Much like in last week's homework, you can characterize the number fluctuations using the grand partition function. Show that when a system is in thermal and diffusive equilibrium with a reservoir, the average number of particles in the system is

$$\overline{N} = \frac{kT}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial \mu}$$

where the partial derivative is taken at fixed temperature and volume. Find a similar formula for $\overline{N^2}$ in terms of a second derivative of $\mathcal{Z}$. Use these results to show that the standard deviation of N is

$$\sigma_N = \sqrt{kT(\partial \overline{N}/\partial \mu)}.$$

Finally, apply this formula to an ideal gas, to obtain a simple expression for σ_N in terms of $\overline{N}$. Discuss your result briefly.

2. In Zak's guest lecture he proved the useful relation $F = -kT \ln Z$ between the Helmholtz free energy and the ordinary partition function. Use an analogous argument to prove that

$$\Phi = -kT \ln \mathcal{Z},$$

where $\mathcal{Z}$ is the grand partition function and Φ is the grand free energy introduced in the last problem of our Hw8.

3. Consider two single-particle states, A and B, in a system of fermions, where $\epsilon_A = \mu - x$ and $\epsilon_B = \mu + x$; that is, level A lies below μ by the same amount that level B lies above μ. Prove that the probability of level B being occupied is the same as the probability of level A being unoccupied. In other words, the Fermi-Dirac distribution is "symmetrical" about the point where $\epsilon = \mu$.

4. Imagine a world in which that there existed a third type of particle with yet another kind of statistics. These particles can share a single-particle state with one other particle of the same type but no more. Thus the number of these particles in any state can be 0, 1, or 2. Derive the distribution function for the average occupancy of a state by particles of this type, and plot the occupancy as a function of the state's energy, for several different temperatures.

5. Consider a degenerate electron gas in which essentially all of the electrons are highly relativistic ($\epsilon \gg mc^2$), so that their energies are $\epsilon = pc$ (where p is the magnitude of the momentum vector).

(a) Modify the derivation given in class to show that for a relativistic electron gas at zero temperature, the chemical potential (or Fermi energy) is given by $\mu = hc(3N/8\pi V)^{1/3}$.
(b) Find a formula for the total energy of this system in terms of N and μ.

6. A **white dwarf** star (see Schroeder's Figure 7.12) is essentially a degenerate electron gas, with a bunch of nuclei mixed in to balance the charge and to provide the gravitational attraction that

holds the star together. In this problem you will derive a relation between the mass and the radius of a white dwarf star, modeling the star as a uniform-density sphere. White dwarf stars tend to be extremely hot by our standards; nevertheless, it is an excellent approximation in this problem to set $T = 0$.

(a) Use dimensional analysis to argue that the gravitational potential energy of a uniform-density sphere (mass M, radius R) must equal

$$U_{\text{grav}} = -(\text{constant})\frac{GM^2}{R},$$

where (constant) is some numerical constant. Be sure to explain the minus sign. The constant turns out to equal 3/5; you can derive it by calculating the work needed to assemble the sphere, shell by shell. [Note: I'm giving you the out of doing this by dimensional analysis, since it is faster, but if you'd prefer to just derive this result, which has come up a few times in this class, feel free to do that too.]
(b) Assuming that the star contains one proton and one neutron for each electron, and that the electrons are nonrelativistic, show that the total (kinetic) energy of the degenerate electrons equals

$$U_{\text{kinetic}} = (0.0086)\frac{h^2 M^{5/3}}{m_e m_p^{5/3} R^2}.$$

The numerical factor can be expressed exactly in terms of π and cube roots and such, but it's not worth it.
(c) The equilibrium radius of the white dwarf is that which minimizes the total energy $U_{\text{grav}}+U_{\text{kinetic}}$. Sketch the total energy as a function of R, and find a formula for the equilibrium radius in terms of the mass. As the mass increases, does the radius increase or decrease? Does this make sense?
(d) Evaluate the equilibrium radius for $M = 2 \times 10^{30}$ kg, the mass of the sun. Also evaluate the density. How does the density compare to that of water?
(e) Calculate the Fermi energy and the Fermi temperature, for the case considered in part (d). Discuss whether the approximation $T = 0$ is valid.
(f) Suppose instead that the electrons in the white dwarf star are highly relativistic. Using the result of the previous problem, show that the total kinetic energy of the electrons is now proportional to $1/R$ instead of $1/R^2$. Argue that there is no stable equilibrium radius for such a star.
(g) The transition from the nonrelativistic regime to the ultrarelativistic regime occurs approximately where the average kinetic energy of an electron is equal to its rest energy, mc^2. Is the nonrelativistic approximation valid for a one-solar-mass white dwarf? Above what mass would you expect a white dwarf to become relativistic and hence unstable?

7. A star that is too heavy to stabilize as a white dwarf can collapse further to form a **neutron star**: a star made entirely of neutrons, supported against gravitational collapse by degenerate neutron pressure. Repeat the steps of the previous problem for a neutron star, to determine the following: the mass-radius relation; the radius, density, Fermi energy, and Fermi temperature of a one-solar-mass neutron star; and the critical mass above which a neutron star becomes relativistic and hence unstable to further collapse.