Homework 3

Due by 10pm on Wednesday, September 23rd, 2020

Reading: Schroeder Chap. 2, sections 2.3-5. Class notes.

1. As an illustration of why it matters which variables you hold fixed when taking partial derivatives, consider the following mathematical example. Let $w = xy$ and $x = yz$.
(a) Write w purely in terms of x and z, and then purely in terms of y and z.
(b) Compute the partial derivatives

$$\left(\frac{\partial w}{\partial x}\right)_y \qquad \text{and} \qquad \left(\frac{\partial w}{\partial x}\right)_z,$$

and show that they are not equal. (Hint: To compute $(\partial w/\partial x)_y$, use a formula for w in terms of x and y, not z. Similarly, compute $(\partial w/\partial x)_z$ from a formula for w in terms of only x and z.)
(c) Compute the other four partial derivatives of w (two each with respect to y and z), and show that it matters which variable is held fixed.

2. A 60-kg hiker wishes to climb to the top of Mt. Ogden, an ascent of 5000 vertical feet (1500 m).
(a) Assuming that she is 25% efficient at converting chemical energy from food into mechanical work, and that essentially all the mechanical work is used to climb vertically, roughly how many bowls of corn flakes (standard serving size 1 ounce, 100 kilocalories) should the hiker eat before setting out?
(b) As the hiker climbs the mountain, three-quarters of the energy from the corn flakes is converted to thermal energy. If there were no way to dissipate this energy, by how many degrees would her body temperature increase?
(c) In fact, the extra energy does not warm the hiker's body significantly; instead, it goes (mostly) into evaporating water from her skin. How many liters of water should she drink during the hike to replace the lost fluids? (At 25°C, a reasonable temperature to assume, the latent heat of vaporization of water is 580 cal/g, 8% more than at 100°C.)

3. Heat capacities are normally positive, but there is an important class of exceptions: systems of particles held together by gravity, such as stars and star clusters.
(a) Consider a system of just two particles, with identical masses, orbiting in circles about their center of mass. Show that the gravitational potential energy of this system is -2 times the total kinetic energy.
(b) The conclusion of part (a) turns out to be true, at least on average, for any system of particles held together by mutual gravitational attraction:

$$\overline{U}_{\text{potential}} = -2\overline{U}_{\text{kinetic}}.$$

Here each $\overline{U}$ refers to the total energy (of that type) for the entire system, averaged over some sufficiently long time period. This result is known as the virial theorem. (For a proof, see Carroll and Ostlie (1996), Section 2.4.) Suppose, then, that you add some energy to such a system and then wait for the system to equilibrate. Does the average total kinetic energy increase or decrease? Explain.
(c) A star can be modeled as a gas of particles that interact with each other only gravitationally.

According to the equipartition theorem, the average kinetic energy of the particles in such a star should be $\frac{3}{2}kT$, where T is the average temperature. Express the total energy of a star in terms of its average temperature, and calculate the heat capacity. Note the sign.
(d) Use dimensional analysis to argue that a star of mass M and radius R should have a total potential energy of $-GM^2/R$, times some constant of order 1.
(e) Estimate the average temperature of the sun, whose mass is 2×10^{30} kg and whose radius is 7×10^8 m. Assume, for simplicity, that the sun is made entirely of protons and electrons.

4. Consider a system of two Einstein solids, A and B, each containing 10 oscillators, sharing a total of 20 units of energy. Assume that the solids are weakly coupled, and that the total energy is fixed.
(a) How many different macrostates are available to this system?
(b) How many different rmcrostates are available to this system?
(c) Assuming that this system is in thermal equilibrium, what is the probability of finding all the energy in solid A?
(d) What is the probability of finding exactly half of the energy in solid A?
(e) Under what circumstances would this system exhibit irreversible behavior?

5. (a) Use python to reproduce the table and graph in Schroeder's Figure 2.4: two Einstein solids, each containing three harmonic oscillators, with a total of six units of energy. Then modify the table and graph to show the case where one Einstein solid contains six harmonic oscillators and the other contains four harmonic oscillators (with the total number of energy units still equal to six). Assuming that all microstates are equally likely, what is the most probable macrostate, and what is its probability? What is the least probable macrostate, and what is its probability?
(b) Use python to produce a table and graph, like those in Figure 2.5 of Schroeder's book, for the case where one Einstein solid contains 200 oscillators, the other contains 100 oscillators, and there are 100 units of energy in total. (Like in his table, you need not show every value, just a sampling of different cases is fine.) What is the most probable macrostate, and what is its probability? What is the least probable macrostate, and what is its probability?

6. **Dry Adiabatic Lapse Rate**—On the last homework you calculated the pressure of earth's atmosphere as a function of altitude, assuming constant temperature. Ordinarily, however, the temperature of the bottommost 10-15 km of the atmosphere (called the troposphere) decreases with increasing altitude, due to heating from the ground (which is warmed by sunlight). If the temperature gradient $|dT/dz|$ exceeds a certain critical value, convection will occur: Warm, low-density air will rise, while cool, high-density air sinks. The decrease of pressure with altitude causes a rising air mass to expand adiabatically and thus to cool. The condition for convection to occur is that the rising air mass must remain warmer than the surrounding air despite this adiabatic cooling.
(a) Show that when an ideal gas expands adiabatically, the temperature and pressure are related by the differential equation

$$\frac{dT}{dP} = \frac{2}{f+2}\frac{T}{P}.$$

(b) Assume that dT/dz is just at the critical value for convection to begin, so that the vertical forces on a convecting air mass are always approximately in balance. Use the result from Hw1 6.(b) to find a formula for dT/dz in this case. The result should be a constant, independent of temperature and pressure, which evaluates to approximately $-10°$ C/km. This fundamental meteorological quantity is known as the **dry adiabatic lapse rate**.

7. **Real (Non-Ideal) Gases**—Even at low density, real gases don't quite obey the ideal gas law. A systematic way to account for deviations from ideal behavior is the virial expansion,

$$PV = nRT\left(1 + \frac{B(T)}{(V/n)} + \frac{C(T)}{(V/n)^2} + \cdots\right),$$

where the functions $B(T), C(T)$, and so on are called the **virial coefficients**. When the density of the gas is fairly low, so that the volume per mole is large, each term in the series is much smaller than the one before. In many situations it's sufficient to omit the third term and concentrate on the second, whose coefficient $B(T)$ is called the second virial coefficient (the first coefficient being 1). Here are some measured values of the second virial coefficient for nitrogen (N_2):

$T(K)$	B (cm^3/mol) of N_2
100	-160
200	-35
300	-4.2
400	9.0
500	16.9
600	21.3

Download and run through the very short "Instructions for Python and Jupyter 2: Defining functions and interacting with graphics notebook" from the computing tab of our website.

(a) Write a python code that, given a temperature and virial coefficient B, computes the second term in the virial equation, $B(T)/(V/n)$. Use this code and the table to compute the second term for nitrogen at atmospheric pressure. Discuss the validity of the ideal gas law under these conditions.
(b) Think about the forces between molecules, and explain why we might expect $B(T)$ to be negative at low temperatures but positive at high temperatures.
(c) Any proposed relation between P, V, and T, like the ideal gas law or the virial equation, is called an equation of state. Another famous equation of state, which is qualitatively accurate even for dense fluids, is the van der Waals equation,

$$\left(P + \frac{an^2}{V^2}\right)(V - nb) = nRT,$$

where a and b are constants that depend on the type of gas. Calculate the second and third virial coefficients (B and C) for a gas obeying the van der Waals equation, in terms of a and b. (Hint: The binomial expansion says that $(1+x)^p \approx 1 + px + \frac{1}{2}p(p-1)x^2$, provided that $|px| \ll 1$. Apply this approximation to the quantity $[1 - (nb/V)]^{-1}$.)
(d) Use python to plot a graph of the van der Waals prediction for $B(T)$, choosing a and b so as to approximately match the data given above for nitrogen. Discuss the accuracy of the van der Waals equation over this range of conditions. (The van der Waals equation is discussed much further in Section 5.3 of our Schroeder text and will be very useful when you get into Molecular Dynamics simulations.)