Homework 5

Due by 10pm on Wednesday, October 7th, 2020

Reading: Schroeder Chap. 3, sections 3.1-3. Class notes.

1. We want to continue practicing our ability to compute multiplicities. Right before Julia's guest lecture I very briefly introduced the two-state paramagnet model, which consists of a long chain of dipoles immersed in a magnetic field $\vec{B}$ (see Fig 2.1 of Schroeder). Each dipole is either aligned (low energy state) or anti-aligned with the magnetic field. If the field points up we call the number of aligned dipoles $N_\uparrow$ and the number of anti-aligned dipoles $N_\downarrow$, we also call the total number of dipoles N.

(a) Without referring to your textbook try to find the multiplicity of the macrostate with $N_\uparrow$ dipoles aligned with an applied magnetic field, $\Omega(N_\uparrow)$. [No problem if you get stuck, then you can refer to Schroeder (pp 52-53), but try it yourself first, as it is important to practice our permutation and combination counting skills.]

(b) Use Stirling's result to find an approximate formula for this multiplicity of a two-state paramagnet. Simplify this formula in the limit $N_\downarrow \ll N$. This result should look very similar to your answer to Schroeder's Problem 2.17 from the last homework; explain why these two systems, in the limits considered, are essentially the same.

Consider a two-state paramagnet with 10^{23} elementary dipoles, with the total energy fixed at zero so that exactly half the dipoles point up and half point down.

(c) How many microstates are "accessible" to this system?

(d) Suppose that the microstate of this system changes a billion times per second. How many microstates will it explore in ten billion years (the age of the universe)?

(e) Is it correct to say that, if you wait long enough, a system will eventually be found in every "accessible" microstate? Explain your answer, and discuss the meaning of the word "accessible."

2. The mathematics of the previous problem can also be applied to a one-dimensional random walk: a journey consisting of N steps, all the same size, each chosen randomly to be either forward or backward. (The usual mental image is that of a drunk stumbling along an alley.)

(a) Where are you most likely to find yourself, after the end of a long random walk?

(b) Suppose you take a random walk of 10,000 steps (say each a yard long). About how far from your starting point would you expect to be at the end?

(c) A good example of a random walk in nature is the diffusion of a molecule through a gas; the average step length is then the mean free path, as computed in Henry's guest lecture (c.f. Section 1.7 of Schroeder). Using this model, and neglecting any small numerical factors that might arise from the varying step size and the multidimensional nature of the path, estimate the expected net

displacement of an air molecule (or perhaps a carbon monoxide molecule traveling through air) in one second, at room temperature and atmospheric pressure. Discuss how your estimate would differ if the elapsed time or the temperature were different. Check that your estimate is consistent with the treatment of diffusion in Schroeder's Section 1.7.

3. This problem gives an alternative approach to estimating the width of the peak of the multiplicity function for a system of two large Einstein solids.

(a) Use Stirling's approximation to show that the multiplicity of an Einstein solid, for any large values of N and q, is approximately

$$\Omega(N,q) \approx \frac{\left(\frac{q+N}{q}\right)^q \left(\frac{q+N}{N}\right)^2}{\sqrt{2\pi q(q+N)/N}}$$

The square root in the denominator is merely large, and can often be neglected. However, it is needed below. (Hint: First show that $\Omega = \frac{N}{q+N}\frac{(q+N)!}{q!N!}$. Do not neglect the $\sqrt{2\pi N}$ in Stirling's approximation.)

(b) Consider two identical Einstein solids, each with N oscillators, in thermal contact with each other. Suppose that the total number of energy units in the combined system is exactly $2N$. How many different macrostates (that is, possible values for the total energy in the first solid) are there for this combined system?

(c) Use the result of part (a) to find an approximate expression for the total number of microstates for the combined system. (Hint: Treat the combined system as a single Einstein solid. Do not throw away factors of "large" numbers, since you will eventually be dividing two "very large" numbers that are nearly equal. Answer: $2^{4N}/\sqrt{8\pi N}$.)

(d) The most likely macrostate for this system is (of course) the one in which the energy is shared equally between the two solids. Use the result of part (a) to find an approximate expression for the multiplicity of this macrostate. (Answer: $2^{4N}/(4\pi N)$.)

(e) You can get a rough idea of the "sharpness" of the multiplicity function by comparing your answers to parts (c) and (d). Part (d) tells you the height of the peak, while part (c) tells you the total area under the entire graph. As a very crude approximation, pretend that the peak's shape is rectangular. In this case, how wide would it be? Out of all the macrostates, what fraction have reasonably large probabilities? Evaluate this fraction numerically for the case $N = 10^23$.

4. On the last homework you found the multiplicity of an ideal gas restricted to "flatland", that is, to a 2D box. Find an expression for the entropy of this two-dimensional ideal gas. Express your result in terms of U, A, and N.

5. (a) Use the Sackur-Tetrode equation to calculate the entropy of a mole of argon gas at room temperature and atmospheric pressure. Why is the entropy greater than that of a mole of helium under the same conditions?

(b) Show that during the quasistatic isothermal expansion of a monatomic ideal gas, the change in entropy is related to the heat input Q by the simple formula

$$\Delta S = \frac{Q}{T}.$$

[Note: This only requires ideas from earlier in the course, not the Sackur-Tetrode equation.] Soon we'll prove that this formula is valid for any quasistatic process. Show, however, that it is not valid for the free expansion process.

6. In Problem 2 you studied a random walk in 1D. Read through and complete the exercises in Python and Jupyter 3: Random Walks.

(a) Once you understand how the random walk works, choose a number of times steps to run it for and compute the displacement x of the walker at the end of this run. Repeat this 10 times, each time saving the total displacement of the walker at the end of the run. After these 10 runs, compute the mean of the square of the 10 displacements, that is, compute $\overline{x^2}$ for these 10 runs. Save that.

Now increase the number of time steps and repeat the whole process 10 mores times. Do this for 7 different numbers of time steps.

Finally, make a plot in Python of the mean squared displacement $\overline{x^2}$ vs. the number of times steps that you used for each of your runs. What do you notice about this plot? Can you think of a way to model the relationship? If so, fit your results to your model and report the parameters of your fit.

Repeat everything you've done here, but now instead of running your code 10 times for each of your chosen number of time steps, run it 1000 times for each of them. Of course, you will not want to do this by hand. Now the model you want to use should be clearer.

(b) Now that you've got a handle on how to do a 1D random walk, figure out how to write your own code for a 2D random walk. Steal anything you want from the 1D code. At each step the walker should be able to step left or right or stay where it is, and step forward or backward or stay where it is.