# Homework 6

Due by 10pm on Wednesday, October 14th, 2020

Reading: Schroeder Chap. 3, sections 3.4-6. Class notes.

1. Using the same method as in Schroeder's text, calculate the entropy of mixing for a system of two monatomic ideal gases, $A$ and $B$, whose relative proportion is arbitrary. Let $N$ be the total number of molecules and let $x$ be the fraction of these that are of species $B$. You should find

$$\Delta S_{\text{mixing}} = -Nk[x \ln x + (1-x)\ln(1-x)].$$

Check that this expression reduces to the one given in the text when $x = 1/2$.

2. The mixing entropy formula derived in the previous problem actually applies to any ideal gas, and to some dense gases, liquids, and solids as well. For the denser systems, we have to assume that the two types of molecules are the same size and that molecules of different types interact with each other in the same way as molecules of the same type (same forces, etc.). Such a system is called an ideal mixture. Explain why, for an ideal mixture, the mixing entropy is given by

$$\Delta S_{\text{mixing}} = k \ln \left[ \binom{N}{N_A} \right],$$

where $N$ is the total number of molecules and $N_A$ is the number of molecules of type $A$. Use Stirling's approximation to show that this expression is the same as the result of the previous problem when both $N$ and $N_A$ are large.

3. Describe a few of your favorite, and least favorite, irreversible processes. In each case, explain how you can tell that the entropy of the universe increases.

4. For either a monatomic ideal gas or a high-temperature Einstein solid, the entropy is given by $Nk$ times some logarithm. The logarithm is never large, so if all you want is an order-of-magnitude estimate, you can neglect it and just say $S \sim Nk$. That is, the entropy in fundamental units is of the order of the number of particles in the system. This conclusion turns out to be true for most systems (with some important exceptions at low temperatures where the particles are behaving in an orderly way). So just for fun, make a very rough estimate of the entropy of each of the following: this book (a kilogram of carbon compounds); a moose (400 kg of water); the sun ($2 \times 10^{30}$ kg of ionized hydrogen).

5. I hesitated to assign this problem initially, but with the Nobel prize this week, I can't stop myself. Enjoy!

A black hole is a region of space where gravity is so strong that nothing, not even light, can escape. Throwing something into a black hole is therefore an irreversible process, at least in the everyday sense of the word. In fact, it is irreversible in the thermodynamic sense as well: Adding mass to a black hole increases the black hole's entropy. It turns out that there's no way to tell (at least from outside) what kind of matter has gone into making a black hole.[1] Therefore, the

[1] This statement is a slight exaggeration. Electric charge and angular momentum are conserved during black hole formation, and these quantities can still be measured from outside a black hole. In this problem I'm assuming for simplicity that both are zero.

entropy of a black hole must be greater than the entropy of any conceivable type of matter that could have been used to create it. Knowing this, it's not hard to estimate the entropy of a black hole.

(a) Use dimensional analysis to show that a black hole of mass $M$ should have a radius of order $GM/c^2$, where $G$ is Newton's gravitational constant and $c$ is the speed of light. Calculate the approximate radius of a one-solar-mass black hole ($M = 2 \times 10^{30}$ kg).

(b) In the spirit of the last problem, explain why the entropy of a black hole, in fundamental units, should be of the order of the maximum number of particles that could have been used to make it.

(c) To make a black hole out of the maximum possible number of particles, you should use particles with the lowest possible energy: long-wavelength photons (or other massless particles). But the wavelength can't be any longer than the size of the black hole. By setting the total energy of the photons equal to $Mc^2$, estimate the maximum number of photons that could be used to make a black hole of mass $M$. Aside from a factor of $8\pi^2$, your result should agree with the exact formula for the entropy of a black hole, obtained (e.g. by Stephen Hawking) through a much more difficult calculation:

$$S_{\rm bh} = k\frac{8\pi^2 GM^2}{hc}.$$

(d) Refining the statement of part (a), it turns out that a black hole's radius is $R_{\rm bh} = 2GM/c^2$ and so the area of the horizon is $A_{\rm bh} = 4\pi R_{\rm bh}^2$. On the other hand, Planck introduced a system of units that has a fundamental length scale, the Planck length: $\ell_P = \sqrt{\hbar G/c^3}$. Express the black hole entropy in terms of $k$, $A_{\rm bh}$, and $\ell_P$. This remarkably simple formula plays an outsized role in our thinking that quantum gravity is relevant to the study of black holes.

(e) Calculate the entropy of a one-solar-mass black hole, and comment on the result.

(f) Use these results to calculate the temperature of a black hole, in terms of its mass $M$. (The energy is $Mc^2$.) Evaluate the resulting expression for a one-solar-mass black hole. Also sketch the entropy as a function of energy, and discuss the implications of the shape of the graph.

6. Use the definition of temperature to prove the zeroth law of thermodynamics, which says that if system $A$ is in thermal equilibrium with system $B$, and system $B$ is in thermal equilibrium with system $C$, then system $A$ is in thermal equilibrium with system $C$. (If this exercise seems totally pointless to you, you're in good company: Everyone considered this "law" to be completely obvious until 1931, when Ralph Fowler pointed out that it was an unstated assumption of classical thermodynamics.)

7. In Schroeder's Section 2.5 he reviews a theorem on the multiplicity of any system with only quadratic degrees of freedom: In the high-temperature limit where the number of units of energy is much larger than the number of degrees of freedom, the multiplicity of any such system is proportional to $U^{Nf/2}$, where $Nf$ is the total number of degrees of freedom. Find an expression for the energy of such a system in terms of its temperature, and comment on the result. How can you tell that this formula for $\Omega$, cannot be valid when the total energy is very small?