Homework 7

Due by 10pm on Wednesday, November 4th, 2020

Reading: Schroeder Chap. 4. Class notes.

1. Consider a monatomic ideal gas that lives at a height z above sea level, so each molecule has potential energy mgz in addition to its kinetic energy.

(a) Show that the chemical potential is the same as if the gas were at sea level, plus an additional term mgz:

$$\mu(z) = -kT \ln\left[\frac{V}{N}\left(\frac{2\pi mkT}{h^2}\right)^{3/2}\right] + mgz.$$

(You can derive this result from either the definition $\mu = -T(dS/dN)_{U,V}$ or the formula $\mu = (dU/dN)_{S,V}$.)

(b) Suppose you have two chunks of helium gas, one at sea level and one at height z, each having the same temperature and volume. Assuming that they are in diffusive equilibrium, show that the number of molecules in the higher chunk is

$$N(z) = N(0)e{-}mgz/kT,$$

in agreement with your previous results on the exponential atmosphere.

2. Suppose you have a *mixture* of gases (such as air, a mixture of nitrogen and oxygen). The **mole fraction** x_i of any species i is defined as the fraction of all the molecules that belong to that species: $x_i = N_i/N_{\text{total}}$. The partial pressure P_i of species i is then defined as the corresponding fraction of the total pressure: $P_i = x_i P$. Assuming that the mixture of gases is ideal, argue that the chemical potential μ_i of species i in this system is the same as if the other gases were not present, at a fixed partial pressure P_i.

3. Previously you computed the entropy of an ideal monatomic gas that lives in a two-dimensional universe. Take partial derivatives with respect to U, A, and N to determine the temperature, pressure, and chemical potential of this gas. (In two dimensions, pressure is defined as force per unit length.) Simplify your results as much as possible, and explain whether they make sense.

4. Recall your work from Hw2, Problem 1, which concerned an ideal diatomic gas taken around a rectangular cycle on a PV diagram. Suppose now that this system is used as a heat engine, to convert the heat added into mechanical work.

(a) Evaluate the efficiency of this engine for the case $V_2 = 3V_1$, $P_2 = 2P_1$.
(b) Calculate the efficiency of an "ideal" engine operating between the same temperature extremes.

5. A power plant produces 1 GW of electricity, at an efficiency of 40% (typical of today's coal-fired plants).

(a) At what rate does this plant expel waste heat into its environment?
(b) Assume first that the cold reservoir for this plant is a river whose flow rate is 100 m^3/s. By

how much will the temperature of the river increase?
(c) To avoid this "thermal pollution" of the river, the plant could instead be cooled by evaporation of river water. (This is more expensive, but in some areas it is environmentally preferable.) At what rate must the water evaporate? What fraction of the river must be evaporated?

6. Problem 4.4. It has been proposed to use the thermal gradient of the ocean to drive a heat engine. Suppose that at a certain location the water temperature is $22°C$ at the ocean surface and $4°C$ at the ocean floor.

(a) What is the maximum possible efficiency of an engine operating between these two temperatures?
(b) If the engine is to produce 1 GW of electrical power, what minimum volume of water must be processed (to suck out the heat) in every second?

7. To get more than an infinitesimal amount of work out of a Carnot engine, we would have to keep the temperature of its working substance below that of the hot reservoir and above that of the cold reservoir by non-infinitesimal amounts (recall that Josh assumed that they were very close). Consider, then, a Carnot cycle in which the working substance is at temperature T_{hw} as it absorbs heat from the hot reservoir, and at temperature T_{cw} as it expels heat to the cold reservoir. Under most circumstances the rates of heat transfer will be directly proportional to the temperature differences:

$$\frac{Q_h}{\Delta t} = K(T_h - T_{hw}) \qquad \text{and} \qquad \frac{Q_c}{\Delta t} = K(T_{cw} - T_c).$$

I've assumed here for simplicity that the constants of proportionality (K) are the same for both of these processes. Let us also assume that both processes take the same amount of time, so the At's are the same in both of these equations.

(a) Assuming that no new entropy is created during the cycle except during the two heat transfer processes, derive an equation that relates the four temperatures T_h, T_c, T_{hw}, and T_{cw}.

(b) Assuming that the time required for the two adiabatic steps is negligible, write down an expression for the power (work per unit time) output of this engine. Use the first and second laws to write the power entirely in terms of the four temperatures (and the constant K), then eliminate T_{cw} using the result of part (a).

(c) When the cost of building an engine is much greater than the cost of fuel (as is often the case), it is desirable to optimize the engine for maximum power output, not maximum efficiency. Show that, for fixed T_h and T_c, the expression you found in part (b) has a maximum value at $T_{hw} = \frac{1}{2}(T_h + \sqrt{T_h T_c})$. (Hint: You'll have to solve a quadratic equation.) Find the corresponding expression for T_{cw}.

(d) Show that the efficiency of this engine is $1 - \sqrt{T_c/T_h}$. Evaluate this efficiency numerically for a typical coal-fired steam turbine with $T_h = 600°C$ and $T_c = 25°C$, and compare to the ideal Carnot efficiency for this temperature range. Which value is closer to the actual efficiency, about 40%, of a real coal-burning power plant?