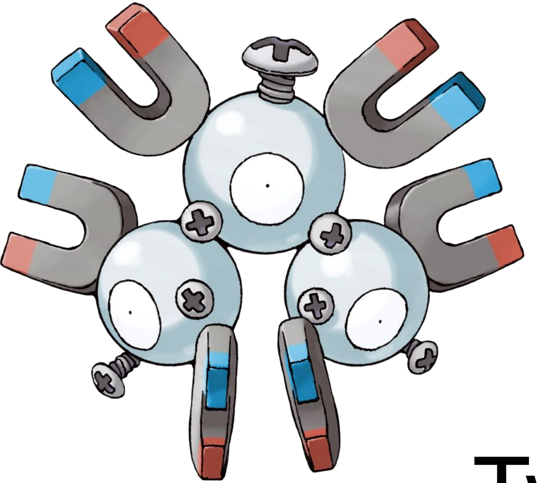
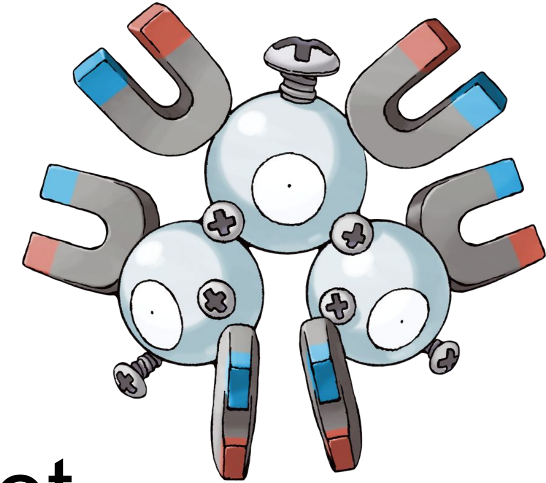
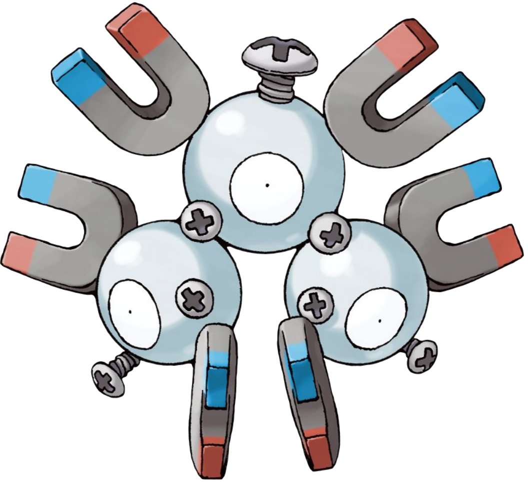
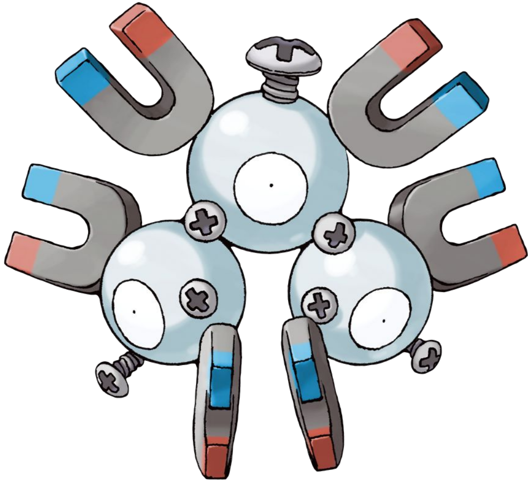




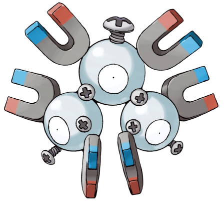
Two State Paramagnet

and Magnetization

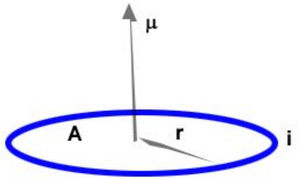




What is the two state paramagnet model?



- Imagine a system of N spin ½ particles (like an electron) immersed in a constant magnetic field pointed in the z-direction. (assume no interactions between particles)
  - We know each particle experiences a torque due to B which will try to align the spin direction of the particle with the field
- This behavior is that of a dipole. One model for a dipole is a current loop



This gives us the definition

$$\vec{\mu} \equiv I\vec{a}$$

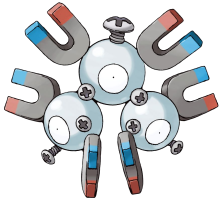
So for a circular loop in xy-plane, with current running counter clockwise,

$$\vec{\mu} = I\pi r^2 \hat{z}$$

If we consider our two state particles to be electrons, μ takes a specific value, called the **Bohr Magneton**.

$$\mu_B \equiv \frac{eh}{4\pi m_e} = 9.274 \times 10^{-24}\ \mathrm{J/T} = 5.788 \times 10^{-5}\ \mathrm{eV/T}.$$

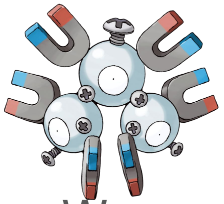
What about a material made up of a ton of dipoles with magnetic moments μ?



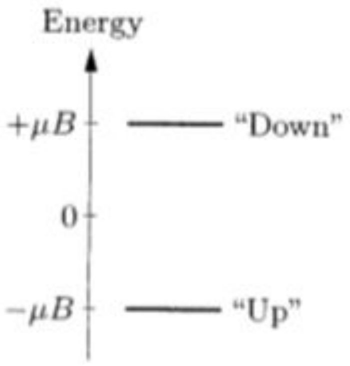
For a material with many dipoles like this, immersed in a magnetic field, we can ask “what is the **net** magnetic dipole moment?”.

This net quantity is called the materials “Magnetization”, and we defined it by,

$$M \equiv \text{magnetic dipole moment per unit volume}$$



With those concepts at hand, let's return to our system of N spin ½ dipoles. We are considering a two state paramagnet, that is, the dipole can be in the “up” energy state or the “down” energy state.



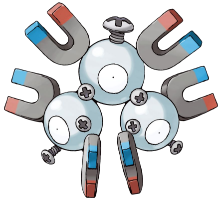
A dipole in a constant magnetic field has energy ±μB. Notice that “up” correlates to aligned with the field, and “down” to anti-aligned. The total number of dipoles then is

$$N = N_{\uparrow} + N_{\downarrow}$$

So, the total energy, U, is then the sum of the energy for all up and down dipoles

$$U = \mu B(N_{\downarrow} - N_{\uparrow}) = \mu B(N - 2N_{\uparrow})$$

$$U = \mu B(N_{\downarrow} - N_{\uparrow}) = \mu B(N - 2N_{\uparrow})$$

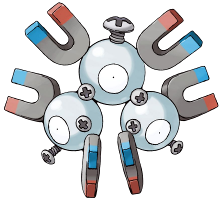


Knowing that the total energy tells us something about the relationship between “up” and “down” states...

We can use our definition of Magnetization (the total magnetic moment of the system), to write

$$M = \mu(N_{\uparrow} - N_{\downarrow}) = -\frac{U}{B}$$

Beautiful relation no? Well, if we remember that U is related to temperature, we can ask ourselves how the magnetization changes (and energy) as the temperature changes…...let's find out!

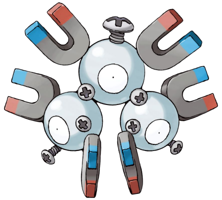


To investigate this question, we first want to know all of the different possible dipole (energy) configurations. That is what the multiplicity tells us. Recall, $\Omega(N_\uparrow) = \frac{N!}{N_\uparrow!(N-N_\uparrow)!} = \frac{N!}{N_\uparrow! N_\downarrow!}$ And using the definition of entropy we find,

$$\begin{aligned}
\frac{S}{k} &= \ln \Omega \\
&= \ln \frac{N!}{N_\uparrow!(N-N_\uparrow)!} \\
&= \ln N! - \ln N_\uparrow! - \ln(N-N_\uparrow)! \\
&\approx N \ln N - N - N_\uparrow \ln N_\uparrow + N_\uparrow - (N-N_\uparrow)\ln(N-N_\uparrow) + (N-N_\uparrow) \\
&= \boxed{N \ln N - N_\uparrow \ln N_\uparrow - (N-N_\uparrow)\ln(N-N_\uparrow)}
\end{aligned}$$

Where we use Stirling's approximation by assuming the number of dipoles in each state is large, thus the total N is large.

$$U = \mu B(N_\downarrow - N_\uparrow) = \mu B(N - 2N_\uparrow)$$



Having this equation of Entropy of our system, we can then use our new definition of temperature continue investigating our question.

Since S(N, N↑) and we can write N↑(U), our temperature equation becomes,

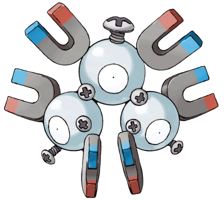
$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,V} = \frac{\partial N_\uparrow}{\partial U}\frac{\partial S}{\partial N_\uparrow}$$

Where this last equality is the chain rule.

We can do this computation in two parts. First, recall the equation for the total energy U of the system. We can solve for N↑(U) to find $N_\uparrow(U) = \frac{1}{2}(N - \frac{U}{\mu B})$

Then, $\frac{\partial N_\uparrow}{\partial U} = \frac{-1}{2\mu B}$.

$$U = \mu B(N_\downarrow - N_\uparrow) = \mu B(N - 2N_\uparrow)$$

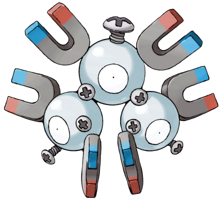


Second, recall the formula for S we derived before. Then,

$$\begin{aligned}
\frac{\partial S}{\partial N_\uparrow} &= k\frac{\partial}{\partial N_\uparrow}(N\ln N - N_\uparrow \ln N_\uparrow - (N - N_\uparrow)\ln(N - N_\uparrow)) \\
&= k\ln(N - N_\uparrow) - \ln N_\uparrow \\
&= k\ln\frac{N - N_\uparrow}{N_\uparrow} \\
&= \boxed{k\ln\frac{N + \frac{U}{\mu B}}{N - \frac{U}{\mu B}}}
\end{aligned}$$

Note that from the third to fourth line, we rewrite N↑ in terms of U as in the previous slide, and simplify.

Having taken both partial derivatives, we can plug them back into our equation for temperature to get,

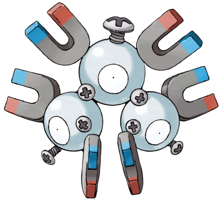


$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_{N,V} = \frac{\partial N_\uparrow}{\partial U}\frac{\partial S}{\partial N_\uparrow} = \frac{-1}{2\mu B}\left(k \ln \frac{N + \frac{U}{\mu B}}{N - \frac{U}{\mu B}}\right) = \frac{k}{2\mu B} \ln \frac{N - \frac{U}{\mu B}}{N + \frac{U}{\mu B}}$$

Eqn. 3.30

Great! We have found a relationship between temperature and energy (and thus magnetization) as we had set out to do. But now we want to isolate U to know how it depends on T.

Solving for U... First, reorganize our last result to get



$$\frac{2\mu B}{kT} = \ln \frac{N - \frac{U}{\mu B}}{N + \frac{U}{\mu B}}$$

Then, exponentiate

$$e^{\frac{2\mu B}{kT}} = \frac{N - \frac{U}{\mu B}}{N + \frac{U}{\mu B}} = \frac{1 - \frac{U}{N\mu B}}{1 + \frac{U}{N\mu B}}$$

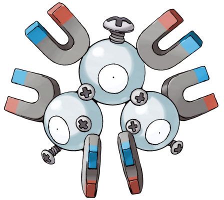
Put together the U terms

$$1 - e^{\frac{2\mu B}{kT}} = \frac{U}{N\mu B}(1 + e^{\frac{2\mu B}{kT}})$$

Solve for U

$$U = N\mu B\left(\frac{1 - e^{\frac{2\mu B}{kT}}}{1 + e^{\frac{2\mu B}{kT}}}\right)$$

We solved for U, but we can simplify.



Recall that $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}$, and so multiplying by minus 1 resembles our

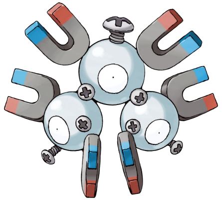
Fraction in our equation for U where x = μB/kT. So finally,

$$U = N\mu B\left(\frac{1-e^{\frac{2\mu B}{kT}}}{1+e^{\frac{2\mu B}{kT}}}\right) = -N\mu B\tanh\left(\frac{\mu B}{kT}\right)$$

Great! This is the simplest form. Now, to know how M depends on T, we plug this into our equation for M. Then,

$$M = -\frac{U}{B} = N\mu\tanh\left(\frac{\mu B}{kT}\right)$$

And so we have done it! We found exactly how M and U depend on T.



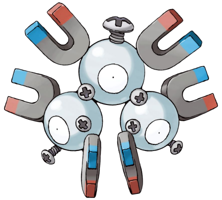
Plugging U back into M...

Recall, $M = \mu(N_{\uparrow} - N_{\downarrow}) = -\frac{U}{B}$

So with our equation for U in terms of T, we get

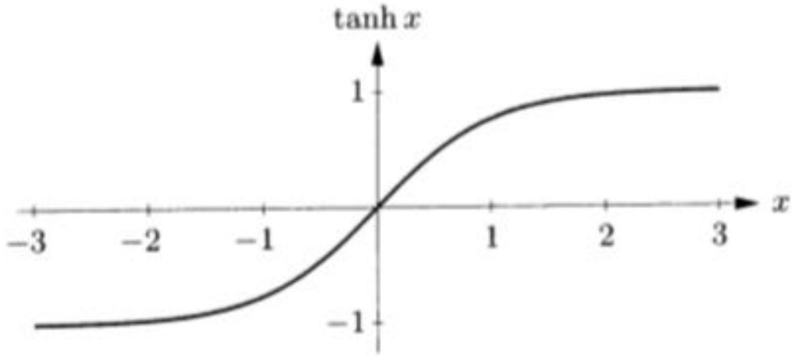
$$M = N\mu \tanh\left(\frac{\mu B}{kT}\right).$$

We see that M behaves like tanh(x). Let's looks at the tanh function to get a better understanding of the physical interpretation.



Remember, $$M = N\mu \tanh\left(\frac{\mu B}{kT}\right).$$

So x = μB/kT



Here is the tanh function. The magnetization of our paramagnet system follows this type of curve as we just found. Notice that as B goes to zero, so does M. Does this make sense? Also, as T goes to infinity, M also goes to zero… how is that?

What about the asymptotic behavior? Well, we see that for small positive T, magnetization becomes maximum (all dipoles pointing up). Similarly, if we flip the direction of the magnetic field, we see the magnetization maximizes again, now with all dipoles pointing down.