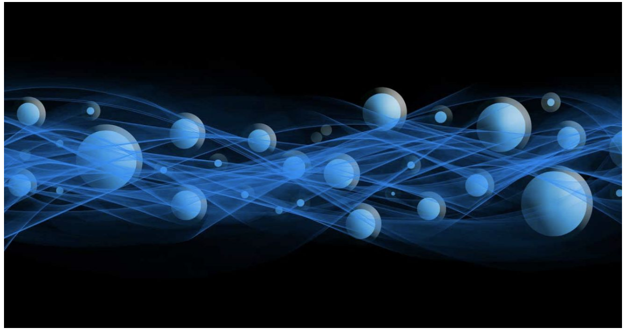
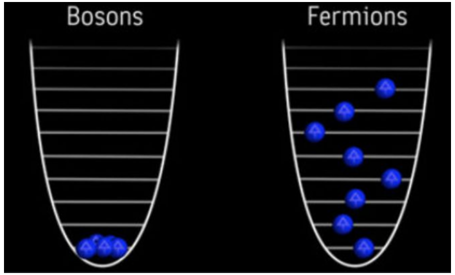
Bose-Einstein Condensation



Boson:

a particle (such as a photon or meson) whose spin quantum number **is** zero or an integral number

These can be packed into the same state



Photons, Phonons:

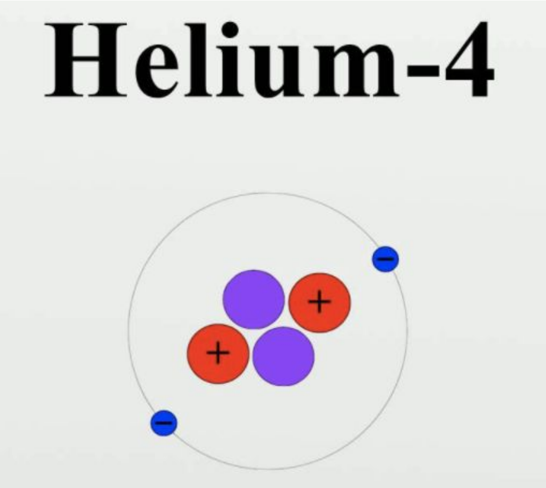
- The Chemical potential μ is zero
- The number of Photons in a given system change frequently

$$N_{total} \neq constant$$

$$\mu = 0$$

"Ordinary Bosons"

For example Helium-4 has bosonic while Helium-3 has Fermionic statistic

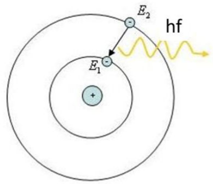


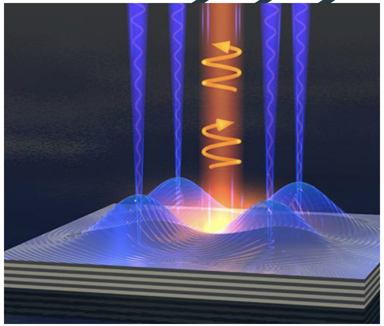
These Bosons are atoms whose fundamental particles add up to an integer spin

The total number of particles is fixed

$$N_{total} = Constant$$

Bose-Einstein Condensation occurs when a gas of Bosons will abruptly condense into the ground state as the Temperature goes below a certain value

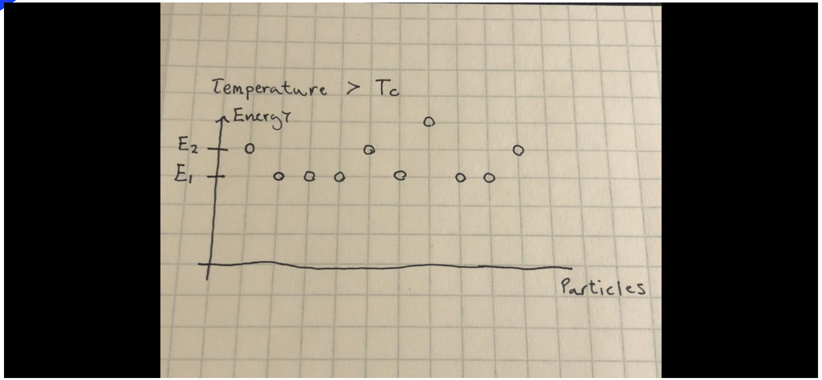


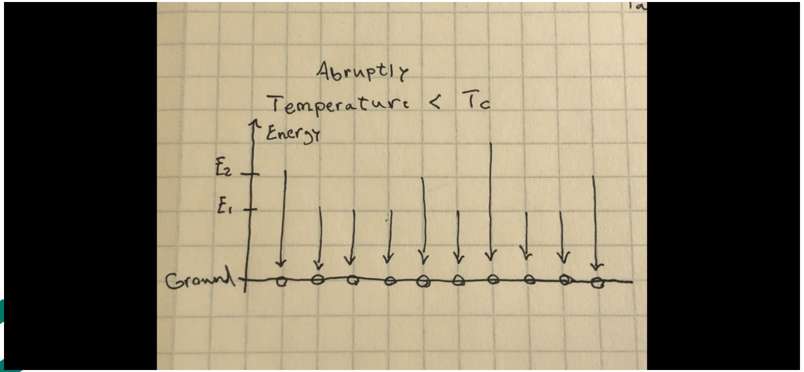


The amount of atoms in a given state is given by the **Bose-Einstein Distribution:**

$$N_0 = \frac{1}{e^{\frac{(\varepsilon_0 - \mu)}{kT}} - 1}$$

For an "Ordinary Boson" the chemical potential (rather than being fixed at zero) is now a nontrivial function of the density and temperature.


Temperature > Tc
Energy
E2
E1
Particles


Abruptly
Temperature < Tc
Energy
E2
E1
Ground

It is simplest to consider the limit $T \to 0$

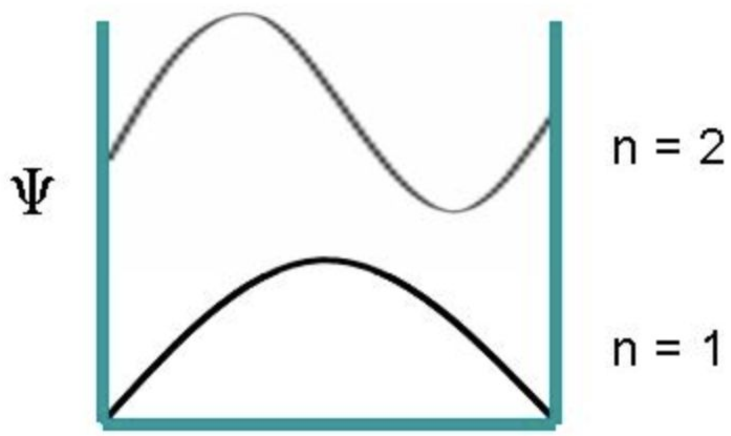
When the temperature is zero Kelvin, all the atoms will be in the lowest-energy available state.
Since arbitrarily many bosons are allowed in any state, this means that every atom will be in the ground state.

(Here when I simply say "State" I mean a single particle state.)

The average number of atoms in a state is given by the Bose-Einstein distribution
So the average number of atoms in the ground state is given by the B.E. distribution when epsilon is equal to the energy of the ground state.

$$N_0 = \frac{1}{e^{\frac{(\varepsilon_0 - \mu)}{kT}} - 1}$$

$$\varepsilon_0 = \textit{The energy of the ground state}$$



To find epsilon we use the same logic that we have used previously of the Debroglie wavelength that fits in a box.

For a box of side lengths L, and volume V^3 the energy of the ground state is:

$$\varepsilon_0 = \frac{h^2}{8mL^2}\left(1^2 + 1^2 + 1^2\right) = \frac{3h^2}{8mL^2} \qquad (7.118)$$

Is there a better way to understand the way that N depends on ε μ and kT?

$$N_0 = \frac{1}{e^{\frac{(\varepsilon_0 - \mu)}{kT}} - 1} \qquad (7.119)$$

A Taylor Series Expansion to simplify the B.E. distribution

$$N_0 = \frac{1}{e^{\frac{(\varepsilon_0 - \mu)}{kT}} - 1}$$

For the Taylor series to work the exponent needs to be close to zero

$$\left(When\ N_0 >> 1 \right)$$

This happens when T is sufficiently low, N will be quite large. In this case the denominator of this expression must be small, which implies the exponential is close to 1.

This means that the exponent $\frac{\varepsilon_0 - \mu}{kT}$ must be small, and so our series should work

The average number of atoms in the ground state is given by 7.120

$$N_0 = \frac{1}{1 + \frac{(\varepsilon_0 - \mu)}{kT} - 1} = \frac{kT}{\varepsilon_0 - \mu} \quad \left(When\ N_0 >> 1 \right) \qquad (7.120)$$

What does this mean?

$$N_0 = \frac{kT}{\varepsilon_0 - \mu}$$

When T=0, we know that the chemical potential μ is equal to ε.

When the temperature is non-zero, but still very small so that most of the atoms are in the ground state, μ is a tiny bit less than ε.

$$When\ T=0,\ \mu = \varepsilon_0$$

$$and\ when\ T>0,\ but\ still\ <<1\quad \mu < \varepsilon_0$$

How low must the temperature be, in order for No to be large

The general condition that determines mew is that the sum of the Bose-Einstein distribution over all states must add up to the total number of atoms, N:

$$N = \sum_{all\ s} \frac{1}{e^{\frac{\in 0 - \mu}{kT}} - 1} \qquad (7.121)$$

To solve this we could keep guessing values of μ until the sum works perfectly and repeat the process for each value of T.

Converting to an integral

$$N = \int_0^{\infty} g(\varepsilon) \frac{1}{e^{\frac{(\varepsilon - \mu)}{kT}} - 1} \qquad (7.122)$$

Valid when kT>>εo

So that the number of terms that contribute significantly to the sum is large.

This integral contains a new function g(ε).

$$N_0 = \frac{1}{e^{\frac{(\varepsilon_0 - \mu)}{kT}} - 1}$$

General Condition for μ

$$N = \sum_{all\ s} \frac{1}{e^{\frac{\in 0 - \mu}{kT}} - 1}$$

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$$N = \int_0^\infty g(\varepsilon) \frac{1}{e^{\frac{(\varepsilon-\mu)}{kT}} - 1}$$

The function $g(\varepsilon)$ is the ***density of states***: The number of single-particle states per unit energy.

- For spin-zero Bosons
- Confined in a box of Volume V

Bear with me here

Same as electrons

$$g(\varepsilon) = \frac{2}{\sqrt{\pi}} \left(\frac{2\pi m}{h^2} \right)^{\frac{3}{2}} V \sqrt{\varepsilon} \quad (7.123)$$

$$N = \int_0^{\infty} g(\varepsilon) \frac{1}{e^{\frac{(\varepsilon - \mu)}{kT}} - 1}$$

Plugging in for μ

$$N = \frac{2}{\sqrt{\pi}} \left(\frac{2\pi m}{h^2} \right)^{\frac{3}{2}} V \int_0^{\infty} \frac{\sqrt{\varepsilon}\, d\varepsilon}{e^{\frac{\varepsilon}{kT}} - 1}$$

With a substitution of variables

$$x = \frac{\varepsilon}{kT}$$

$$N = \frac{2}{\sqrt{\pi}} \left(\frac{2\pi \mathrm{mkT}}{h^2} \right)^{\frac{3}{2}} V \int_0^{\infty} \frac{\sqrt{x}\, dx}{e^x - 1}$$

$$\frac{2}{\sqrt{\pi}}$$

This integral over x gives 2.315;
Combining this number with the factor $\frac{2}{\sqrt{\pi}}$

We get the formula

$$N = 2.612\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} V \qquad (7.125)$$

This formula can only work for a specific Temperature that we will call Tc.

$$N = 2.612\left(\frac{2\pi mkTc}{h^2}\right)^{\frac{3}{2}} V \qquad (7.126)$$

Or

$$kT_c = 0.527\left(\frac{h^2}{2\pi m}\right)\left(\frac{N}{V}\right)^{\frac{2}{3}}$$

Now to prove that we made a good choice of μ

$$N = 2.612\left(\frac{2\pi mkT}{h^2}\right)^{\frac{3}{2}} V$$

Eq. (7.125)

$$N = \int_0^\infty g(\varepsilon)\,\frac{1}{e^{\frac{(\varepsilon-\mu)}{kT}} - 1}$$

Eq. (7.122)

At T>Tc the chemical potential must be significantly less than 0.

A negative value of μ will yield a result for N that is smaller than the right hand side of equation 7.125.

At temperatures lower than Tc it gets a bit more tricky

The bottom line is this:

At temperatures higher than Tc the chemical potential is negative and essentially all of the atoms are in the excited state. At temperatures lower than Tc, the chemical potential is very close to zero and the number of atoms in excited states is given by equation 7.127

$$N_{excited} = \left(\frac{T}{T_c}\right)^{\frac{3}{2}} N \text{ when } T < Tc$$