Today

I. Last Time
II. Mathematical Preliminaries: The Gamma Function & Sterling's Approximation
III. Return to the Idea Gas: Complete Our Derivation of the Sackur-Tetrode Equation

I. Phase space (x, p), in old quantum theory an observable $A(x, p)$ will have quantized values if its level curves capture finite areas in the phase space.

We also found that the number of quantum states is given by the area captured in phase space divided by h.

Fundamental postulate of Stat. Mech.: All accessible microstates are equally probable when a system is in thermal equilibrium.

II. Recall the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}.$$

Taking the derivative with respect to a

$$\int_{-\infty}^{\infty} -x^2 e^{-ax^2} dx = \sqrt{\pi}(-\frac{1}{2}a^{-3/2}).$$

We want to leverage this idea to understand factorials in a much richer context:

$$\int_{0}^{\infty} e^{-ax} dx = \frac{1}{a} = a^{-1}.$$

Derivative with respect to a is

$$\int_{0}^{\infty} xe^{-ax} dx = 1 \cdot a^{-2}, \int_{0}^{\infty} x^2 e^{-ax} dx = 2a^{-3}, \ldots.$$

The pattern is exactly that of the factorial, setting $a = 1$, gives

$$n! = \int_{0}^{\infty} x^n e^{-x} dx$$

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Taking this integral as a definition of the factorial, we realized that we can define non-integer factorials. This allows us to extend the definition of $r!$ off of the integers.

With this realization of a much broader context for the definition of the factorial people re-named it the "Gamma function":

$$\Gamma(n+1) \equiv \int_0^\infty x^n e^{-x} dx.$$

It's a quick check that I leave to you that

$$\Gamma(n+1) = n\Gamma(n) = n \int_0^\infty x^{n-1} e^{-x} dx.$$

II. The Sterling Approximation

The Sterling approximation gives the value of the factorial for large inputs:

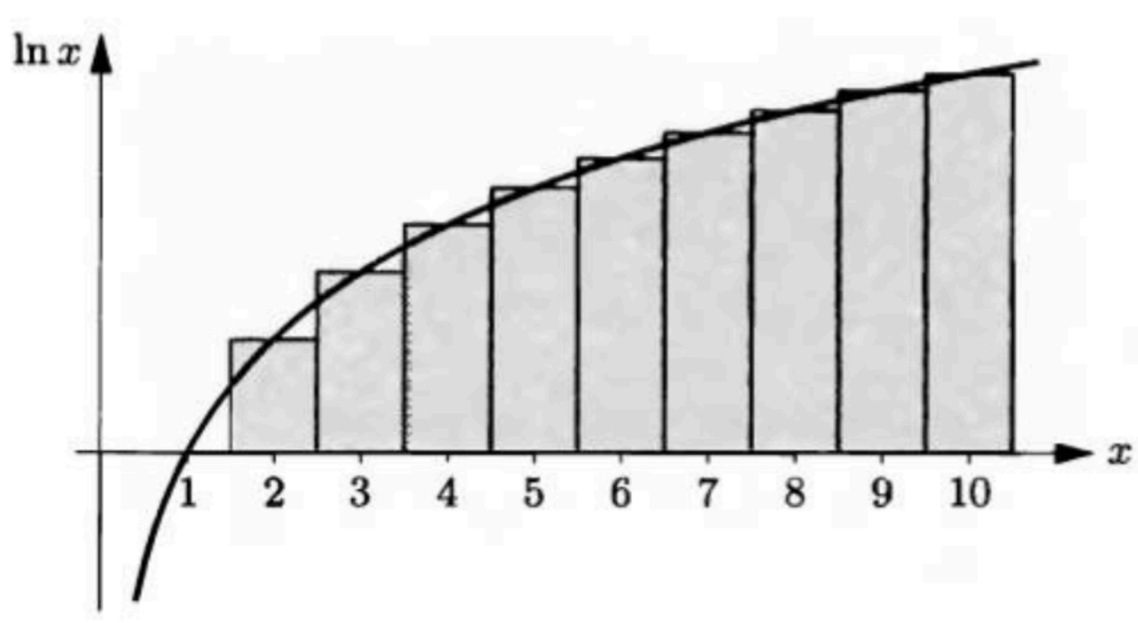
$$n! \approx n^n e^{-n}\sqrt{2\pi n}.$$

If we take the natural log of both sides we get

$$\ln n! \approx (n \ln n) - n.$$

Let's derive this quickly, consider

$$\ln n! = \ln(n \cdot (n-1)\cdots 2 \cdot 1) = \ln n + \ln(n-1) + \cdots + \ln 2 + \ln 1$$



II. The Sterling Approximation

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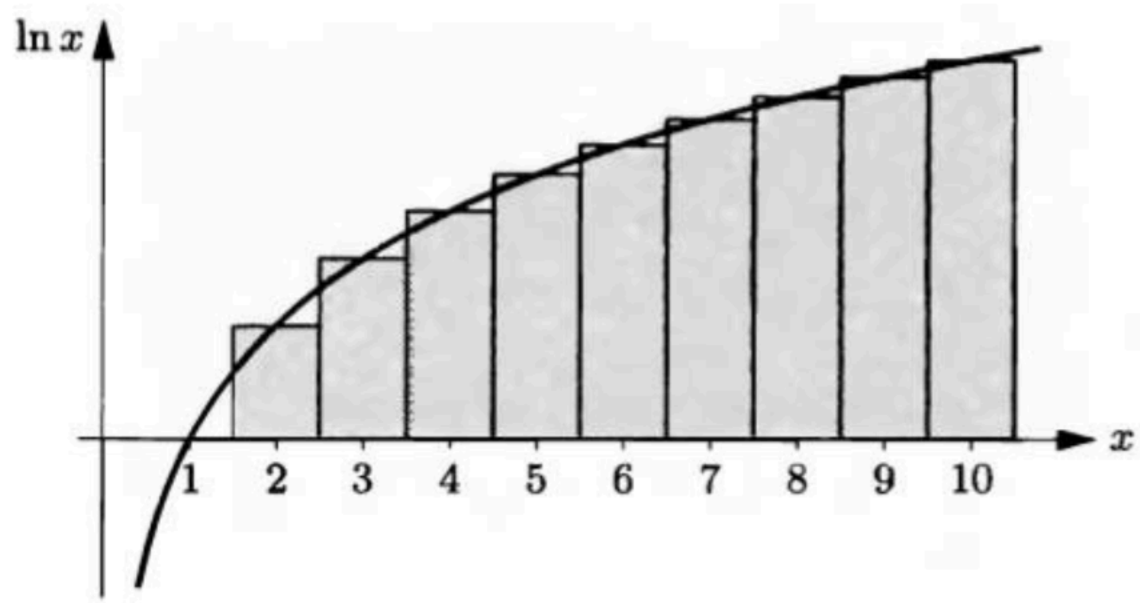
Approximate this sum by the integral,

$$\ln n! \approx \int_0^n \ln(x)dx = (x \ln x - x)\Big|_0^n = n \ln n - n.$$



II. Deriving the surface "area" of a sphere in any number of dimensions

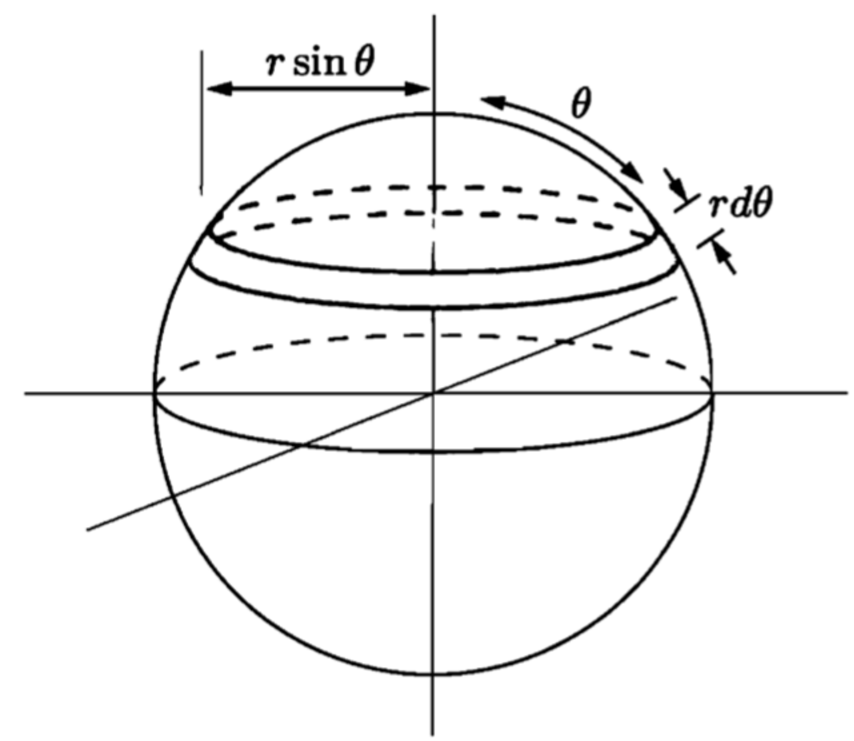
The result for a *d*-dimensional hypersphere

$$A_d(r) = \frac{2\pi^{d/2}}{(\frac{d}{2}-1)!}r^{d-1} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})}r^{d-1}.$$

The case $d = 2$ is for the circle living in the plane, this gives $A_2(r) = 2\pi r$. You can also check $d = 3$, I'll leave that to you. Neat idea is to assume the general case works At first and check that it works for larger d:

$$A_3(r) = \int_0^\pi A_2(r\sin\theta) r d\theta = 2\pi r^2 \int_0^\pi \sin\theta d\theta$$

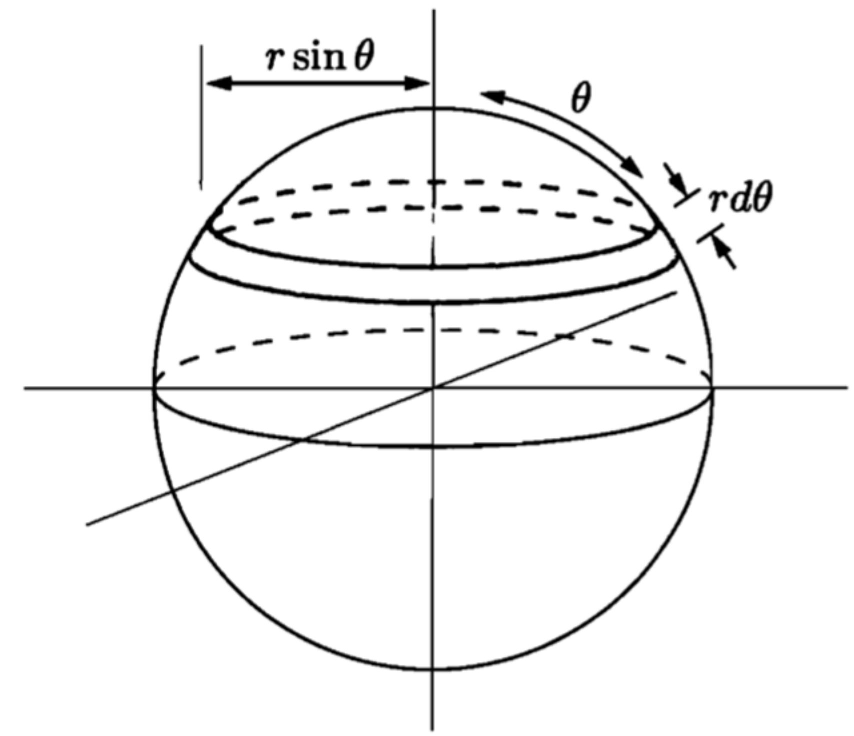
$$= 4\pi r^2$$



II. Now we proceed by induction

$$A_d(r) = \int_0^\pi A_{d-1}(r\sin\theta) r d\theta = \int_0^\pi \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})}(r\sin\theta)^{d-2} r d\theta$$

$$= \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} r^{d-1} \int_0^\pi (\sin\theta)^{d-2} d\theta$$



Note that

$$\int_0^\pi (\sin\theta)^n d\theta = \frac{\sqrt{\pi}\,\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+1)},$$

So we have

$$A_d(r) = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})} r^{d-1} \cdot \frac{\sqrt{\pi}\,\Gamma(\frac{d-2}{2}+\frac{1}{2})}{\Gamma(\frac{d-2}{2}+1)} = \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} r^{d-1}. \checkmark$$