Today

I. Office Hour Juggle: today 4:30-5:30pm
II. Last Time
III. Return to the Idea Gas: Complete Our Derivation of the Sackur-Tetrode Equation

I. Derived the surface "area" of the hypersphere

$$A_d(r) = \frac{2\pi^{d/2}}{(\frac{d}{2} - 1)!} r^{d-1}.$$

In odd numbers of dimensions we have to take fraction factorials, to do this we use the Gamma function

$$\Gamma(\tfrac{d}{2}) = (\tfrac{d}{2} - 1)!$$

The definition of the $\Gamma$ function is

$$\Gamma(n + 1) = \int_0^\infty x^n e^{-x} dx.$$

I. We also reviewed a particularly useful approximation, Sterling's approximation

$$\ln n! = n \ln n - n.$$

We had considered a single particle in a box of volume $V$, with a fixed total energy $U$. What's the multiplicity of this particle classically? Infinite! Because all the points are distinguishable classically.

Fortunately, Quantum Mechanics saves us here! It cuts the number of states down to a finite number! Here the multiplicity is

$$\Omega_1 = \frac{V \cdot V_p}{h^3}$$

II. Let's try to see how to generalize the formula $\Omega_1 = \dfrac{V \cdot V_p}{h^3}$ to any number of particles. First consider two particles

$$\Omega_2 = \frac{V^2}{h^6} \times \text{allowed momentum space.}$$

What is the allowed momentum space for two particles?

$$p_{1x}^2 + p_{1y}^2 + p_{1z}^2 + p_{2x}^2 + p_{2y}^2 + p_{2z}^2 = 2mU.$$

What is this equation an equation of? This one is the equation of what Mathematicians would call the 5-sphere. In Schroeder's notation this is $d = 6$. From our last class we know that this hypersphere's hyper volume is $A_6(\sqrt{2mU})$. There's one remaining subtlety, if the particles are indistinguishable, then we should really have

$$\Omega_2 = \frac{1}{2}\frac{V^2}{h^6} \times \text{allowed momentum space.}$$

II. $\Omega_2 = \frac{1}{2}\frac{V^2}{h^6} \times$ allowed momentum space.

$$A_d(r) = \frac{2\pi^{d/2}}{(\frac{d}{2}-1)!} r^{d-1}$$

Let's try the $N$ particle case:

$$\Omega_N = \frac{1}{N!}\frac{V^N}{h^{3N}} \times \text{area of momentum hypersphere}$$

Let's figure out the allowed momentum space: what's the dimension of the hypersphere of interest? This is a $(3N-1)$-dimenstional hypersphere. This lives in a space of $d = 3N$. Then we have

$$\Omega_N = \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{2\pi^{3N/2}}{\left(\frac{3N}{2}-1\right)!}(\sqrt{2mU})^{3N-1} \approx \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{2\pi^{3N/2}}{\left(\frac{3N}{2}\right)!}(\sqrt{2mU})^{3N}$$

A nice way of looking at this formula is to suppress the numerical factors

$$\Omega(U, V, N) = f(N) V^N U^{3N/2}.$$

II. Entropy of an Ideal Gas:

$$\Omega_N = \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{2\pi^{3N/2}}{\left(\frac{3N}{2}\right)!}(\sqrt{2mU})^{3N}.$$

We take k ln of each side to get the entropy:

$$S = k\ln\Omega_N = k\ln\left(\frac{1}{N!}\frac{V^N}{h^{3N}}\frac{2\pi^{3N/2}}{\left(\frac{3N}{2}\right)!}(\sqrt{2mU})^{3N}\right)$$

$$= k\ln\left(\frac{1}{N!}\frac{1}{\left(\frac{3N}{2}\right)!}\right) + k\ln\left(\frac{V^N}{h^{3N}}2\pi^{3N/2}(\sqrt{2mU})^{3N}\right)$$

$$= k\ln\left(\frac{1}{N!}\frac{1}{\left(\frac{3N}{2}\right)!}\right) + Nk\ln\left(2V\left(\frac{2\pi mU}{h^2}\right)^{3/2}\right)$$

$$
\begin{aligned}
S = k \ln \Omega_N &= k \ln \left( \frac{1}{N!} \frac{V^N}{h^{3N}} \frac{2\pi^{3N/2}}{\left(\frac{3N}{2}\right)!} (\sqrt{2mU})^{3N} \right) \\
&= k \ln \left( \frac{1}{N!} \frac{1}{\left(\frac{3N}{2}\right)!} \right) + k \ln \left( \frac{V^N}{h^{3N}} 2\pi^{3N/2} (\sqrt{2mU})^{3N} \right) \\
&= k \ln \left( \frac{1}{N!} \frac{1}{\left(\frac{3N}{2}\right)!} \right) + Nk \ln \left( 2V \left( \frac{2\pi mU}{h^2} \right)^{3/2} \right) \\
&= k \ln \frac{1}{N!} + k \ln \frac{1}{\left(\frac{3N}{2}!\right)} + \cdots \\
&= kN \left( -\ln N - \frac{3}{2} \ln \frac{3N}{2} + \frac{5}{2} \right) + Nk \ln \left( 2V \left( \frac{2\pi mU}{h^2} \right)^{3/2} \right)
\end{aligned}
$$

$$S = kN\left(-\frac{3}{2}\ln\frac{3N}{2} + \frac{5}{2}\right) + Nk\ln\left(2\frac{V}{N}\left(\frac{2\pi mU}{h^2}\right)^{3/2}\right)$$

$$= kN\left(\frac{5}{2}\right) + Nk\ln\left(2\frac{V}{N}\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2}\right)$$

$$= Nk\left[\ln\left(2\frac{V}{N}\left(\frac{4\pi mU}{3Nh^2}\right)^{3/2}\right) + \frac{5}{2}\right]$$

Upon further investigation, I didn't make any algebra errors in this derivation… see if you can figure out what actually happened. [Answer on next slide.]

Upon further investigation, I didn't make any algebra errors in this derivation. Rather, when we first got the multiplicity, I didn't make one of the approximations that Schroeder does. He approximates the multiplicity this way:

$$\Omega_N = \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{2\pi^{3N/2}}{\left(\frac{3N}{2}-1\right)!}(\sqrt{2mU})^{3N-1} \approx \frac{1}{N!}\frac{V^N}{h^{3N}}\frac{\pi^{3N/2}}{\left(\frac{3N}{2}\right)!}(\sqrt{2mU})^{3N},$$

Where he also drops the factor of $2$ in the numerator. This is a cheat that isn't that well motivated at this point. He's doing it because he knows the answer and wants to get it exactly right. We'll derive this formula again later in the course and won't have to make any such cheat!