Today

I. Last Time
II. Sharpness of the Multiplicity
III. Entropy of an Ideal Gas

I. Finished the calculation of the multiplicity of an ideal gas of N particles:

$$\Omega_N \approx \frac{1}{N!} f(N) V^N U^{3N/2} = \frac{1}{N!} \frac{V^N}{h^{3N}} \frac{\pi^{3N/2}}{\left(\frac{3N}{2}\right)!} (\sqrt{2mU})^{3N}.$$

Then we computed the entropy

$$S = k \ln \Omega_N = Nk \left[\ln \left(\frac{V}{N} \left(\frac{4\pi m U}{3Nh^2} \right)^{3/2} \right) + \frac{5}{2} \right].$$

II. On the homework you studied the multiplicity of an Einstein solid in the large temperature limit: $q \gg N$. In that limit you found

$$\Omega \approx \left(\frac{eq}{N}\right)^N.$$

Let's consider a large system made up of two subsystems A and B. Then, what is the multiplicity for the big system?

$$\Omega(q_A, q_B) = \left(\frac{eq_A}{N}\right)^N \left(\frac{eq_B}{N}\right)^N = \left(\frac{e}{N}\right)^{2N} (q_A q_B)^N.$$

Let $q = q_A + q_B$, this allows us to study things as a function of q_A. Then, let's consider

$$q_A = \frac{q}{2} + x, \text{ and } q_B = \frac{q}{2} - x.$$

Putting these into the multiplicity we have

$$\Omega(q_A, q_B) = \left(\frac{eq_A}{N}\right)^N \left(\frac{eq_B}{N}\right)^N = \left(\frac{e}{N}\right)^{2N} \left(\left(\frac{q}{2}\right)^2 - x^2\right)^N$$

II.Putting these into the multiplicity we have

$$\Omega(q_A, q_B) = \left(\frac{eq_A}{N}\right)^N \left(\frac{eq_B}{N}\right)^N = f(N)\left(\left(\frac{q}{2}\right)^2 - x^2\right)^N.$$

Then the log is

$$\ln \Omega = \ln f + N \ln\left(\left(\frac{q}{2}\right)^2 - x^2\right) = \ln f + N \ln\left[\left(\frac{q}{2}\right)^2\left(1 - \left(\frac{2x}{q}\right)^2\right)\right]$$

$$\approx \ln f + N\left[\ln\left(\frac{q}{2}\right)^2 - \left(\frac{2x}{q}\right)^2\right]$$

Now, let's exponentiate to get back to the multiplicity:

$$\Omega = \left(\frac{e}{N}\right)^{2N} e^{N\ln(q/2)^2} e^{-N(2x/q)^2} = \Omega_{\max} e^{-N(2x/q)^2}.$$

A gaussian has the form $g(x) = Ae^{-ax^2} = Ae^{-\frac{x^2}{2\sigma^2}}$.

II. Now, let's exponentiate to get back to the multiplicity:

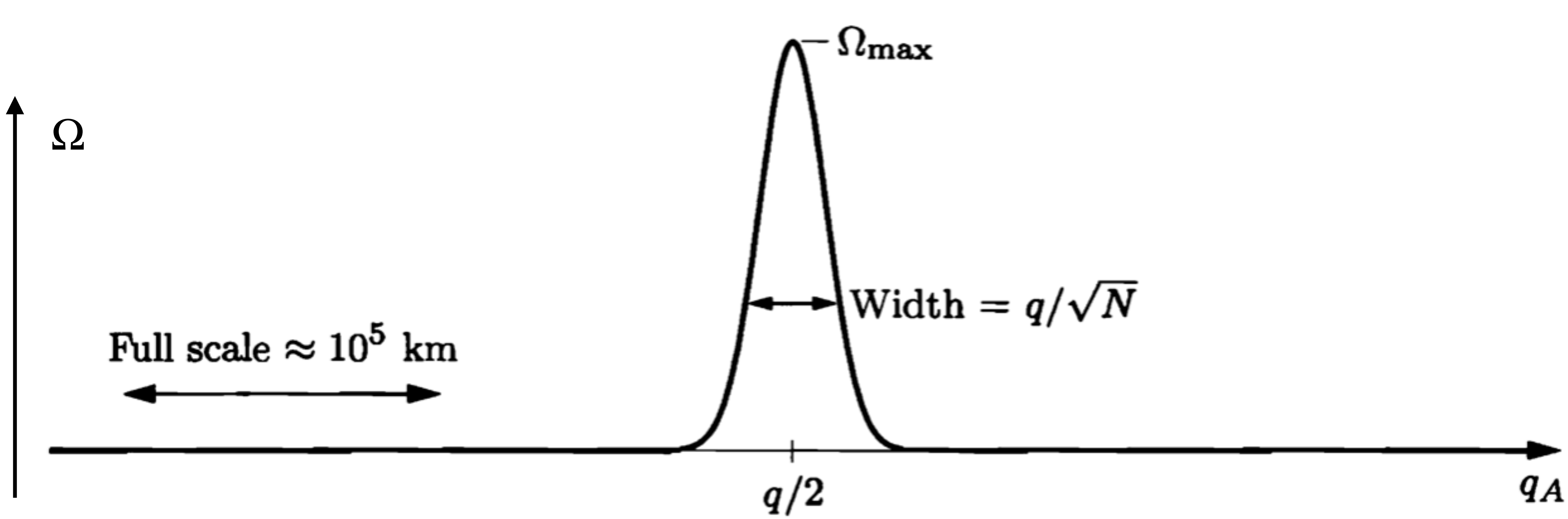
$$\Omega = \left(\frac{e}{N}\right)^{2N} e^{N\ln(q/2)^2} e^{-N(2x/q)^2} = \Omega_{\max} e^{-N(2x/q)^2}.$$

A gaussian has the form $g(x) = Ae^{-ax^2} = Ae^{-\frac{x^2}{2\sigma^2}}$.

We can read off the standard deviation then; it is

$$\sigma = \frac{q}{2\sqrt{2N}}.$$

(Note that Schroeder uses the $1/e$ value instead, which gives instead a half width=$q/(2\sqrt{N})$.)

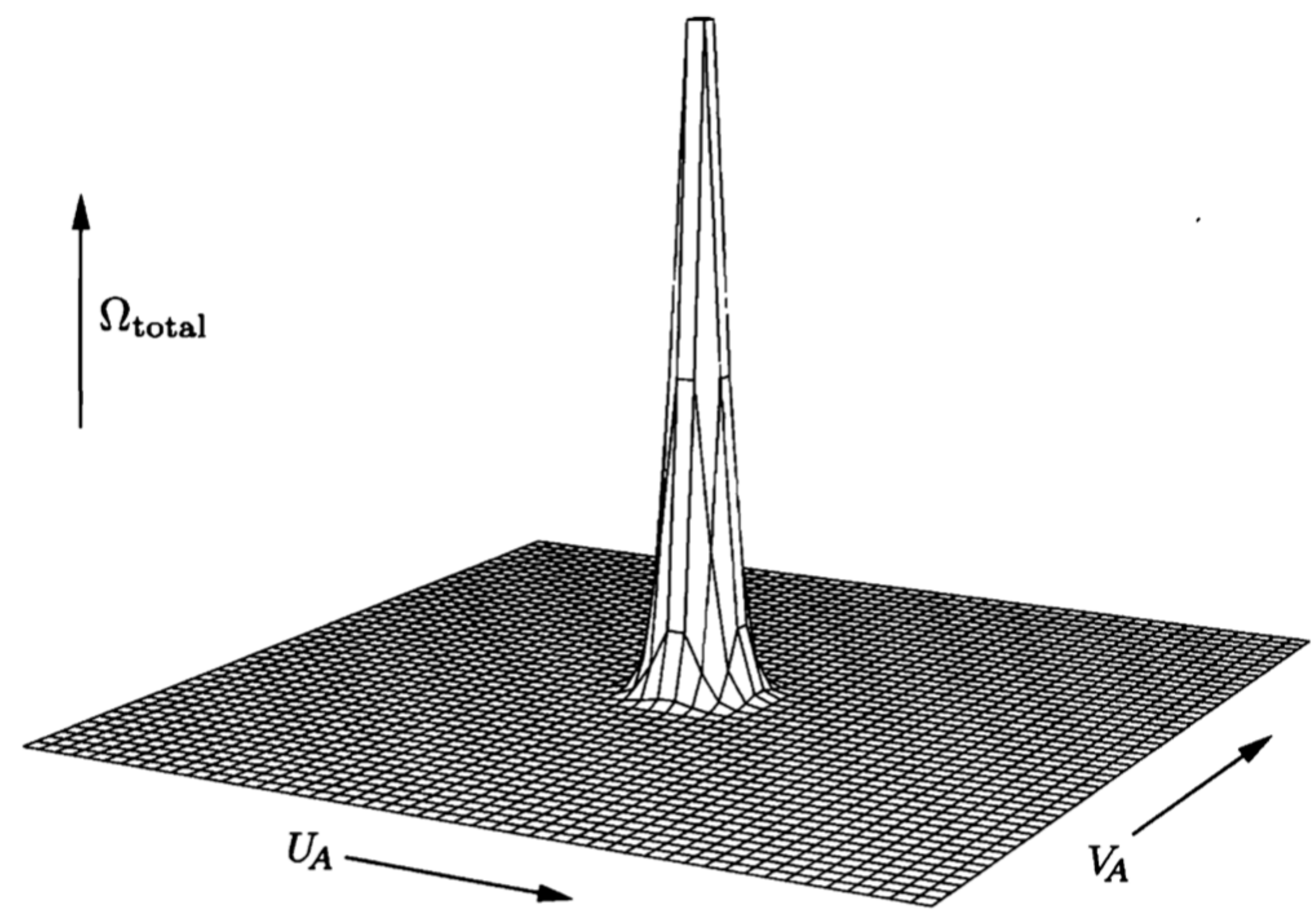


II. A very similar analysis of the ideal gas

$$\Omega_{\text{total}} = [f(N)]^2 (V_A V_B)^N (U_A U_B)^{3N/2},$$

$$\text{width in energy} = \frac{U_{\text{total}}}{\sqrt{3N/2}}$$

$$\text{width in volume} = \frac{V_{\text{total}}}{\sqrt{N}}$$



III. We discussed the following ideas in class, but I didn't write notes out during that conversation. That said, it may be helpful to have them written out too:

A (large) macroscopic system in equilibrium will be found in the macrostate that has the largest multiplicity. This is due to the large probability of that state, not because the evolution requires this state.

This is a way of casting the 2nd law of thermodynamics and can be summarized as "multiplicity tends to increase."

As you know, we usually cast this law in terms of the closely related entropy, $S = k \ln \Omega$, and say:
A (large) macroscopic system in equilibrium will be found in the macrostate that has the largest entropy. Or, entropy tends to increase.