Today

I. Last Time
II. Wrap up Identical Particle Distribution Functions
III. Degenerate Fermi Gas

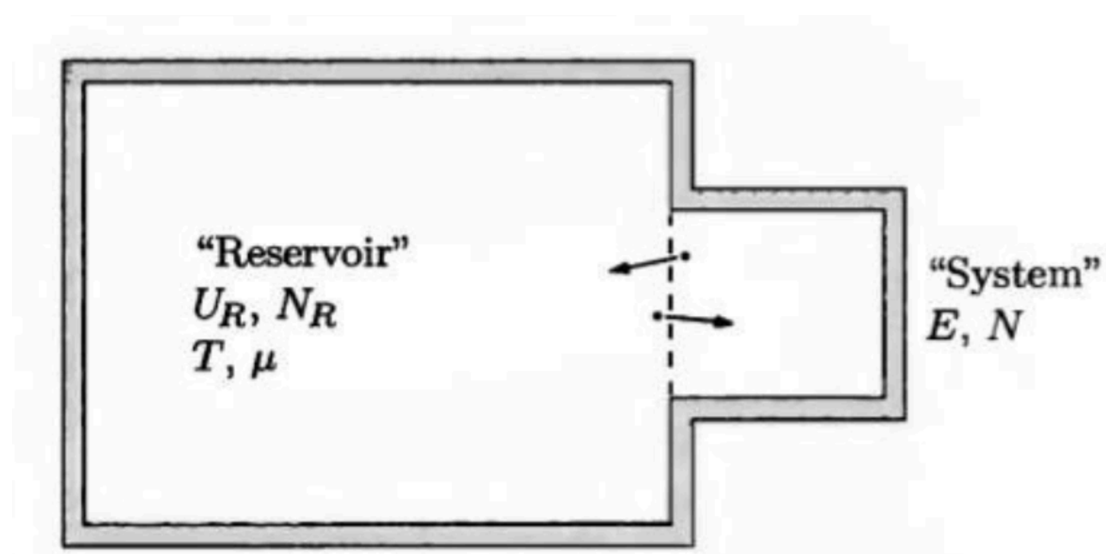
I. Spencer gave a guest lecture on Gibbs factors and the grand partition function

$$e^{-(E(s)-\mu N(s))/kT}.$$

The grand partition function is sum over these Gibbs factors

$$\mathcal{Z} = \sum_s e^{-(E(s)-\mu N(s))/kT}.$$

The setup for these is a system in contact with a constant temperature reservoir that can also exchange particles.

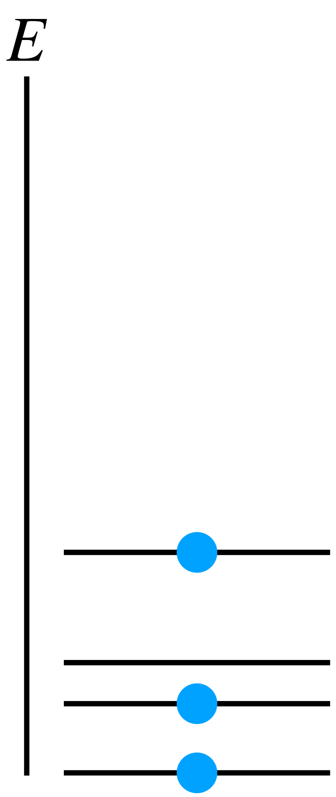


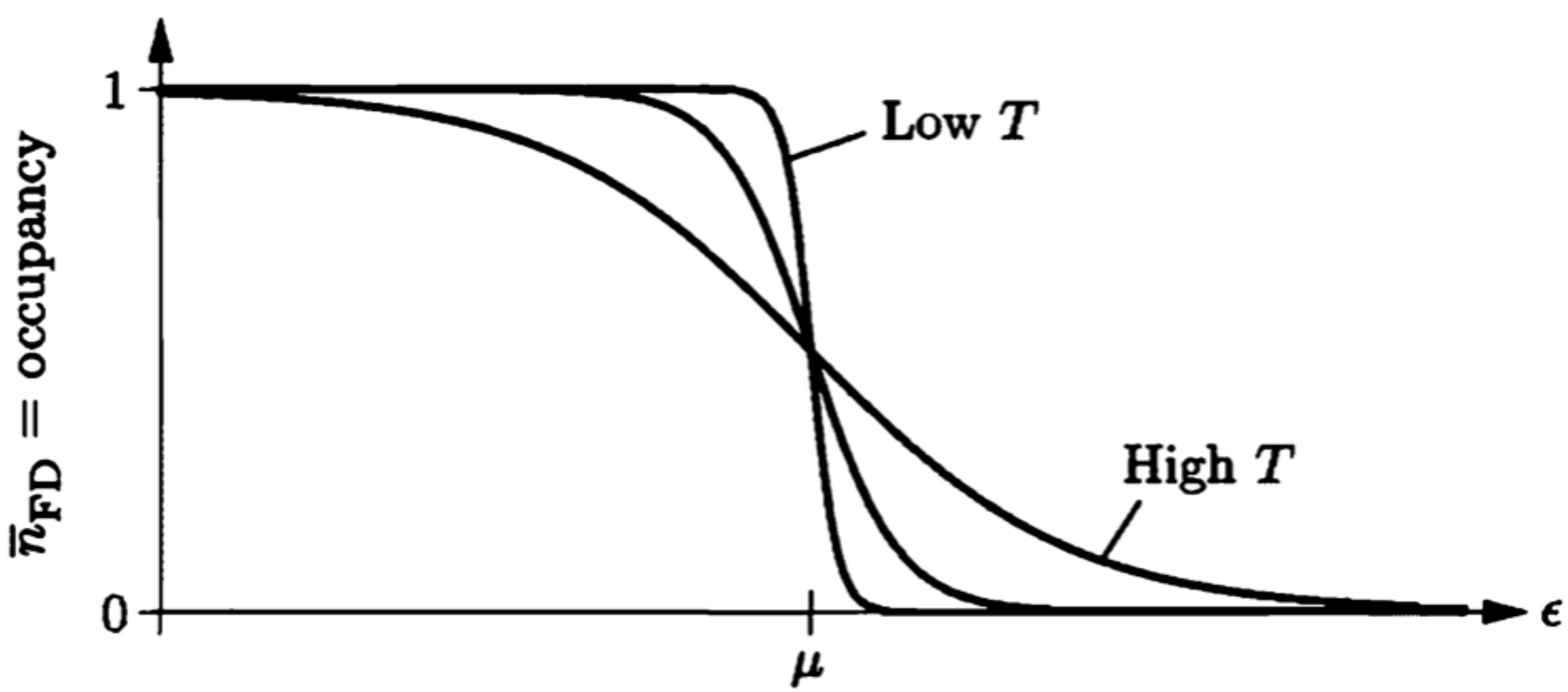
I. We found the grand partition function for fermions

$$\mathcal{Z} = 1 + e^{-(\epsilon-\mu)/kT}.$$

From this we found the "occupancy", that is, the average number of particles in the state under consideration. We found

$$\overline{n}_{FD} = \frac{1}{e^{(\epsilon-\mu)/kT} + 1}.$$





I. Let's turn to the grand partition function for bosons

$$\mathcal{Z} = \sum_n e^{-n(\epsilon-\mu)/kT} = 1 + e^{-(\epsilon-\mu)/kT} + e^{-2(\epsilon-\mu)/kT} + \cdots$$

$$= 1 + e^{-(\epsilon-\mu)/kT} + \left(e^{-(\epsilon-\mu)/kT}\right)^2 + \cdots.$$

$$= \frac{1}{1 - e^{-(\epsilon-\mu)/kT}}$$

Notice that this sum only converges if the term being raised to a power has a magnitude less than 1. Here this amounts to the requirement that ϵ be greater than μ!

Once again we compute occupancy via

$$\bar{n} = \sum_n nP(n) = 0P(0) + 1P(1) + \cdots .$$

Let's introduce the shorthand $x \equiv (\epsilon - \mu)/kT$…

II. We have: $\mathcal{Z} = \dfrac{1}{1 - e^{-(\epsilon-\mu)/kT}}$

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$$\bar{n}_{BE} = \sum_n n\frac{e^{-nx}}{\mathcal{Z}} = \frac{-1}{\mathcal{Z}}\sum_n \frac{\partial}{\partial x}e^{-nx} = -\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial x} = \frac{1}{e^{(\epsilon-\mu)/kT} - 1}$$

If we had stuck with Boltzmann statistics we would have found

$P(s) = \dfrac{1}{Z_1}e^{-\epsilon/kT}$, $\bar{n}_{Boltzmann} = NP(s) = \dfrac{N}{Z_1}e^{-\epsilon/kT} = e^{\mu/kT}e^{-\epsilon/kT} = e^{-(\epsilon-\mu)/kT}$.

On the homework you're showing that $\mu = -kT\ln(Z_1/N)$

II. This is nicely summarized graphically.

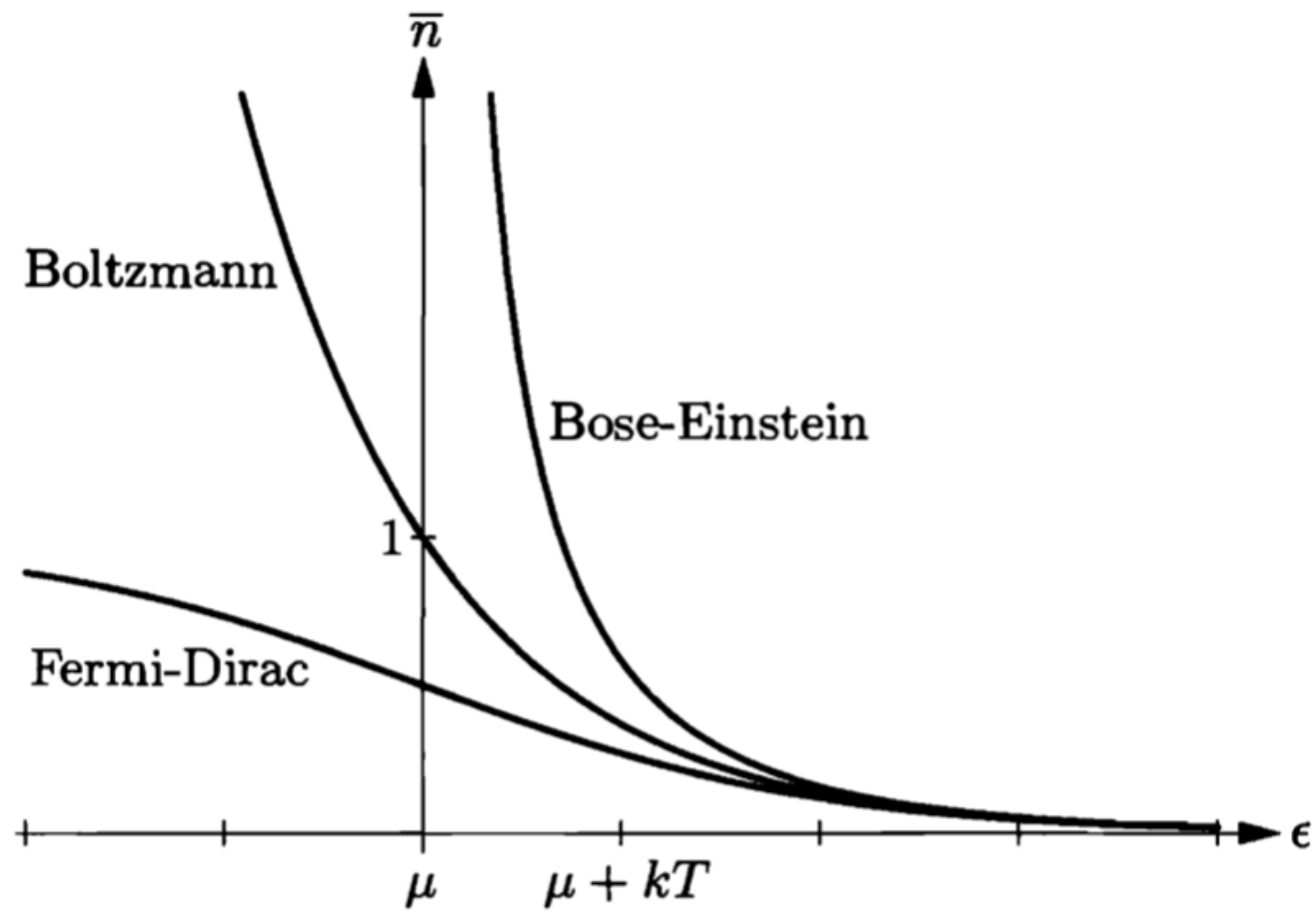


Figure 7.7. Comparison of the Fermi-Dirac, Bose-Einstein, and Boltzmann distributions, all for the same value of μ. When $(\epsilon - \mu)/kT \gg 1$, the three distributions become equal.

III. Degenerate Fermi Gases

To fix a context let's imagine the electrons in a metal as our fermions.

$$v_Q = \left(\frac{h}{\sqrt{2\pi mkT}}\right)^3 = (4.3\text{ nm})^3 \quad \text{(Room temperature gas of electrons)}$$

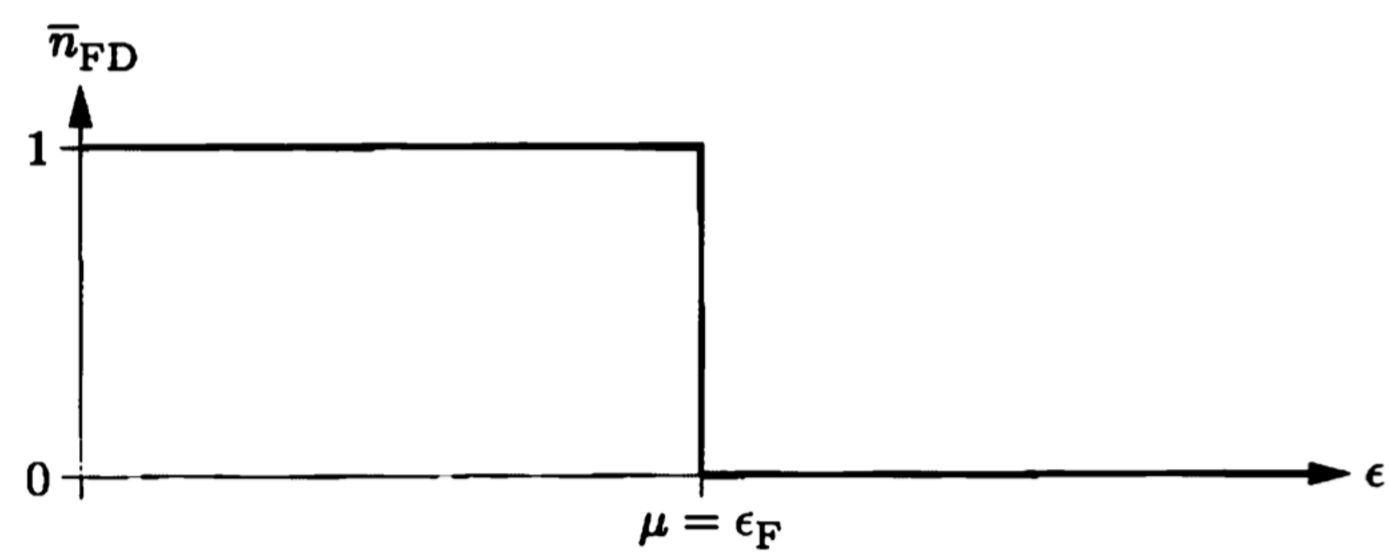
Compare an electron per atom, which gives (0.1 nm)^3, which much smaller.

This shows that we are well away from the dilute limit and we should think about the Fermi-Dirac statistics. In that case, why not think of this as a zero temperature limit. We'll genuinely justify this after the fact.

III. Degenerate Fermi Gases

This shows that we are well away from the dilute limit and we should think about the Fermi-Dirac statistics. In that case, why not think of this as a zero temperature limit. We'll genuinely justify this after the fact. In the zero temperature limit the chemical potential becomes the deciding factor as to whether a state is occupied or not and we call it the "Fermi energy" of the system:

$$\epsilon_F \equiv \mu(T = 0).$$



III. Fermi energy:

$$\epsilon_F \equiv \mu(T = 0).$$

Let's again fix our attention on a cubical box of side length L (we're think of it as a chunk of metal). Recall that free particles in such a box have

$\lambda_n = \dfrac{2L}{n}$, and $p_n = \dfrac{h}{\lambda_n} = \dfrac{hn}{2L}$, but this time let's take into account the 3D nature of the box

$p_x = \dfrac{hn_x}{2L}$, $p_y = \dfrac{hn_y}{2L}$, $p_z = \dfrac{hn_z}{2L}$. The corresponding energies

$$\epsilon = \frac{|\overrightarrow{p}|^2}{2m} = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2).$$