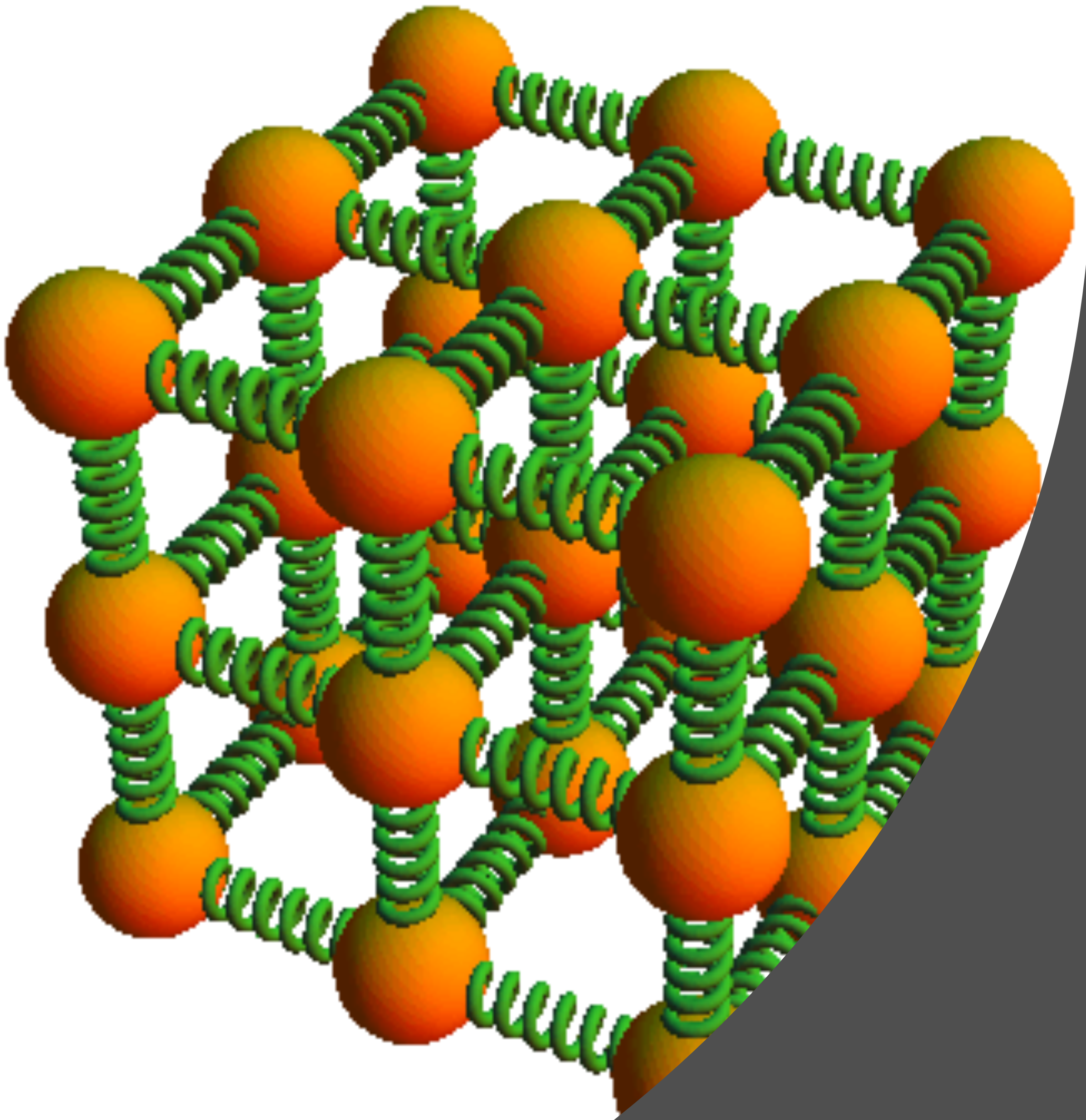


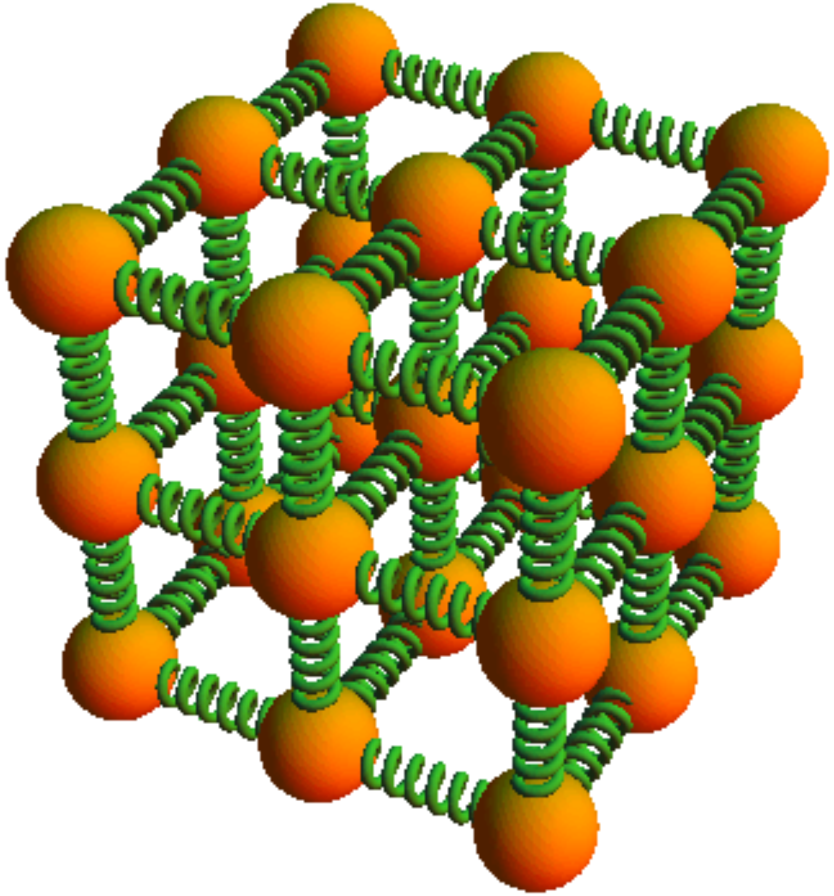
EINSTEIN SOLID

Guest Lecture

OUTLINE

I. What is an Einstein Solid
II. How do we count?
III. Counting Multiplicities of Einstein Solid
IV. General Formula
V. Entropy??

WHAT IS AN EINSTEIN SOLID?



Simple model for the microscopic interactions between the atoms of a solid.

Imagine that these interactions can be described as if they were quantum harmonic oscillators.

We know that for objects in a stable equilibrium, the harmonic oscillator is a good approximation for small disturbances

HOW DO WE COUNT

This is a bit of an aside but we need to understand these tools for counting before we can use them for the Einstein solid.

Lets suppose Matt has q grapes, and N students he could give them to.

For this first scenario, lets assume, he won’t give any student more then one grape.

What we want to know is how many different possible people could he give grapes to?

HOW DO WE COUNT

Lets think, for the first grape Matt has N possible students he could give a grape.

For the second grape, he has N-1, because he won't give any student the more then 1 grape.

This gives us N*(N-1) possible options, however, we don't care which order Matt gives the students the grapes, so we have over counted each options once.

This gives N*(N-1)/2.

We can do this again for the third grape, Matt has N-2 options, but we don't care what order so we over counted by a factor of 3 this time. Giving N*(N-1)*(N-2)/(3*2).

HOW DO WE COUNT

Okay, so how long can we keep doing this? Well, until Matt runs out of grapes.

Lets suppose that he only has 5 grapes. Then we could do this process 2 more times.

Giving: N*(N-1)*(N-2)*(N-3)*(N-4)/(5*4*3*2)

Great! Now we know how many possibilities if Matt has only 5 grapes, but what if he has q grapes?

HOW DO WE COUNT

Giving: N*(N-1)*(N-2)*(N-3)*(N-4)/(5*4*3*2)

Great! Now we know how many if Matt has only 5 grapes, but what if he has q grapes?

Lets rewrite the numerator in terms of factorials. We can see that it should be related to N! .

However, we stop multiplying when Matt runs out of grapes. We can account for this by dividing by (N-q)!

In essence, our numerator is equivalent to

$N!/(N-q)!$

Now, we know we have double counted because we don’t really care which order he gives the grapes to his students.

To account for this we divided by the number of ways 5 students could have been arranged, or 5! And if Matt had q grapes we would divide by q!

HOW DO WE COUNT

Phew, that's a lot of counting. Lets put it all together.

All in all we have:

$$N!/(N-q)!q! = \binom{N}{q}$$

We call this the choose function, and it tells us how many combinations there are when we choose q of N objects.

FROM GRAPES TO ENERGY

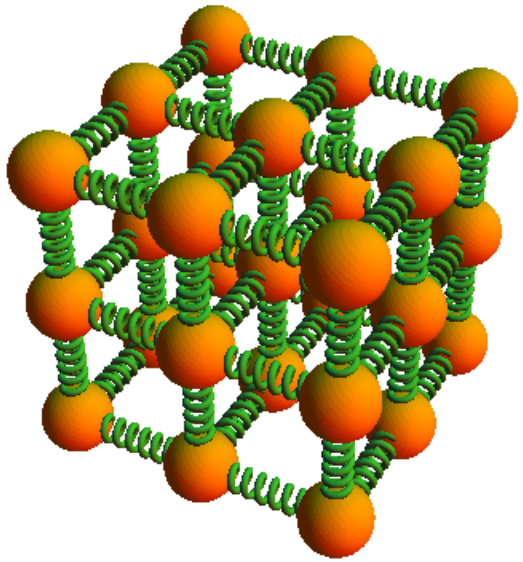
Now let's consider the scenario in which Matt can give more then one grape to his N students, and also that (slightly unrealistically) he can't tell the difference between any of his students

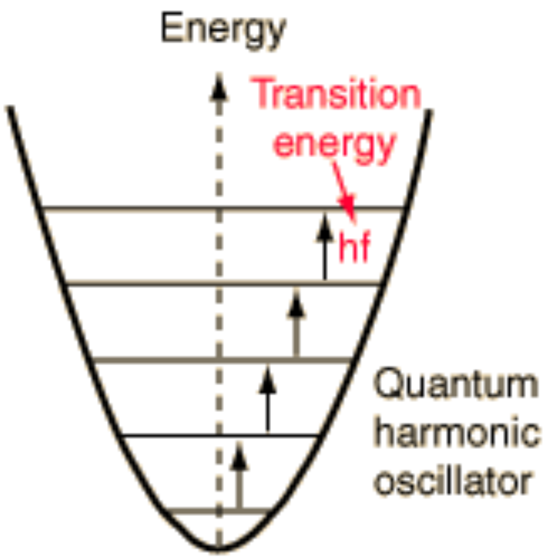
So more concretely, lets say Matt has 3 grapes, and 3 students, Guillermo, Nathalie and Zak. Matt only knows he has 3 grapes and 3 students but wants to know how many different ways he could have given them.

So for example, he could have given Zak 3 grapes (and 0 to Nathalie and Guillermo), 1 to each, or 2 to Nathalie and 1 to Guillermo.

He doesn't know the difference between the situations, but wants to know how many ways there are.

FROM GRAPES TO ENERGY





Okay, so this scenario is very similar to that of our Einstein solid.

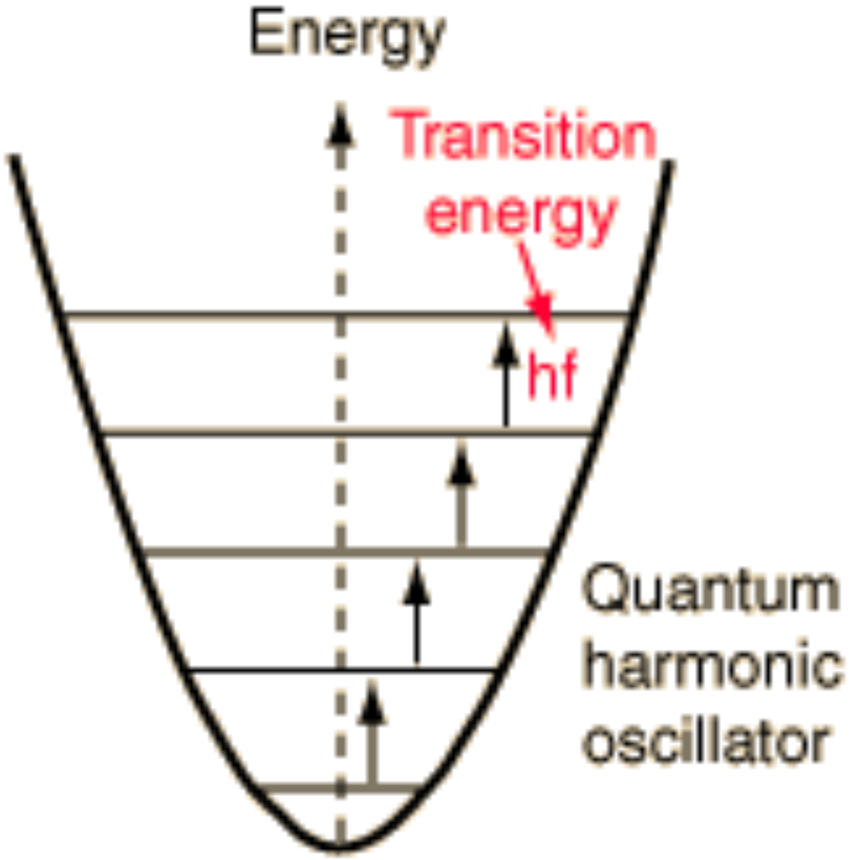
We know our goal in this course is to connect the micro and macroscopic worlds, so lets first remind ourselves of the microscopic world of the Einstein Solid.

The way we are imagining the Einstein Solid is as a collection of quantum oscillators.

Because these are quantum oscillators, they can only take quantized amounts of energy. In other words, I can’t put any energy, I have to move in particularly sized steps.

Luckily, for the quantum oscillator, all the steps have the same size, hf, where h is Planck's constant and f is the natural frequency of the oscillator.

COUNTING MULTIPLICITES



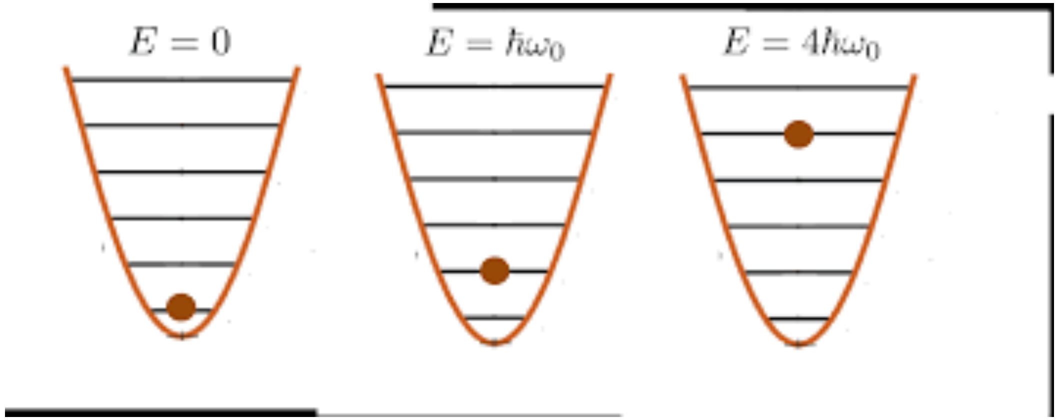
Okay, so now lets see how this is related to Matt and his grapes.

Lets suppose we have N oscillators. This means, that we have N/3 atoms, because each atom can oscillate in 3 dimensions.

Our discrete energy levels, mean that instead of a continuous variable, energy can be counted. We can ask the question of how many hfs are there rather then how much energy.

So if we were to describe the whole microstate entirely, we would need to tell you how many hf's there are in each oscillator, and specify which ones.

COUNTING MULITPLICTIES

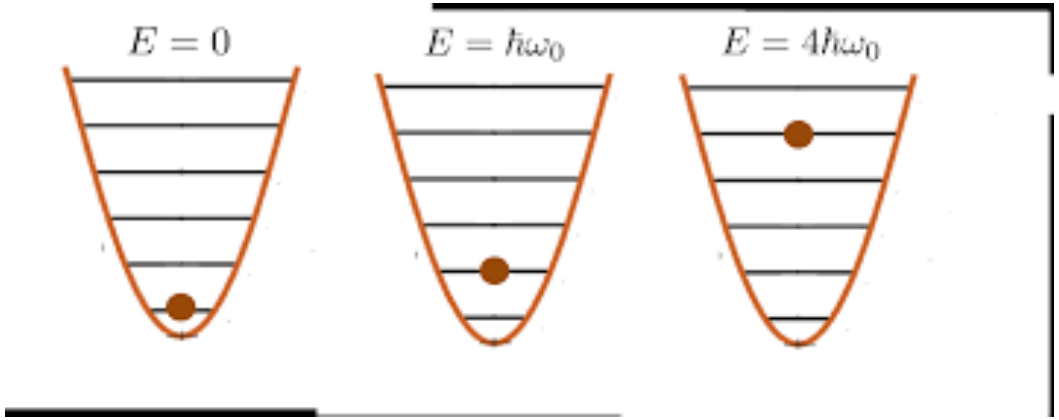


Okay, so now we have N oscillators, each with some energy, an integer multiple of hf.

Suppose we look at the macroscopic variable, total energy, U=qhf, where q is some integer. Just like Matt only knew the total number of grapes, here we only know the total amount of energy.

Technically, the ground state of the quantum harmonic oscillator still has an energy of (1/2)hf, but it increases by hf each time giving excited states of (3/2)hf, (5/2)hf… By comparing everything to the ground state, we can take this to be integer multiples of hf.

COUNTING MULITPLICTIES



Then, we could ask how many possible microstate configurations (how many combinations of students) are there given some number of oscillators, N, and some energy defined by q. We call this the multiplicity.

This means we would like to define a function Ω(N,q) that tells us the multiplicity at a given N and q.

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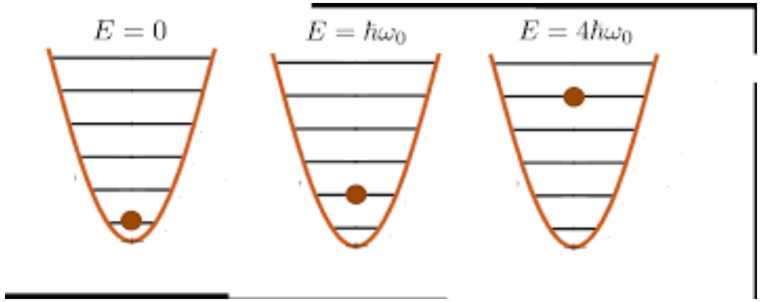
COUNTING MULTIPLICITIES



Oscillator:	#1	#2	#3
Energy:	0	0	0
	1	0	0
	0	1	0
	0	0	1
	2	0	0
	0	2	0
	0	0	2
	1	1	0
	1	0	1
	0	1	1
	3	0	0
	0	3	0
	0	0	3
	2	1	0
	2	0	1
	1	2	0
	0	2	1
	1	0	2
	0	1	2
	1	1	1

To understand more specifically how multiplicity works, lets take a simple case where N=3 and see how the multiplicity changes with q.

If q is 0?

Then all 3 oscillators are in the ground state. There is only 1 way to do this.

If q is 1?

Then there are 3 ways to achieve this energy.

If q is 2?

6

If q is 3?

10

We should expect some kind of choose function, because we are looking for combinations. But its clear we aren't quite in the first scenario.

So, lets look at the general formula and see if we can make some sense of it.

COUNTING MULTIPLICITIES

$$\Omega(N, q) = \binom{q + N - 1}{q}$$

The formula for the number of multiplicities is shown on the left.

Lets set the problem up in the following way, to understand this formula.

On the left, the lines separate our harmonic oscillators. This means we will have N-1 lines

The dots represent the energy units each oscillator has. This means we will have q dots.

COUNTING MULTIPLICITIES

$$\Omega(N, q) = \binom{q + N - 1}{q}$$

• | • • • | | • • • •

So, if we have a total of q+N-1 symbols.

If wherever we place the dots and lines, we are representing some microstate.

The number of microstates that correspond to energy q are thus all possible ways to take q+N-1 symbols and choose q of them to be dots.

A FEW TEST CASES

$$\Omega(N,q) = \binom{q+N-1}{q}$$

Recall from earlier, $(\_N@q) = N!/(N-q)!q!$.

Lets find the multiplicity for N=3,q=3. We did this case explicitly by writing out the all the possibilities, but do we get the same answer here?

Now lets try for the case draw with lines and dots? What is N? What is q? and what is the multiplicity for this macrostate?

$$\Omega(N,q) = \binom{q+N-1}{q}$$

ENTROPY?

3	0	0
0	3	0
0	0	3
2	1	0
2	0	1
1	2	0
0	2	1
1	0	2
0	1	2
1	1	1

Okay, so now that we have calculated some of these multiplicities lets talk about why.

You may have noticed that some of our microstates were more common then others.

Take the N=3, q=3 case. There were a lot more states with 2 energies in one oscillator and 1 in another then any other microstate. The smallest one had only 1 (1,1,1) and whereas 2,1 had 6 possibilities.

When we increase N and q, the disparity between the smallest set of microstates and the largest increases dramatically.

This means that some microstates are simply more likely, and at high N and q, overwhelmingly so.

This is how we think about the idea that entropy always increase. It doesn't always, its just overwhelmingly likely to increase.

YAYY

We now understand Einstein's model of a solid in which each atom's motion is described by 3 quantum oscillators.

We saw the choose function and how to use it.

We calculated the multiplicity of a given solid if we know the number of atoms, and its energy, with the formula

$$\Omega(N, q) = \binom{q + N - 1}{q}$$

We thus understand the connection between the microstates, the energy of each particular oscillator and the macrostate total energy.

And even connected this loosely to Entropy.