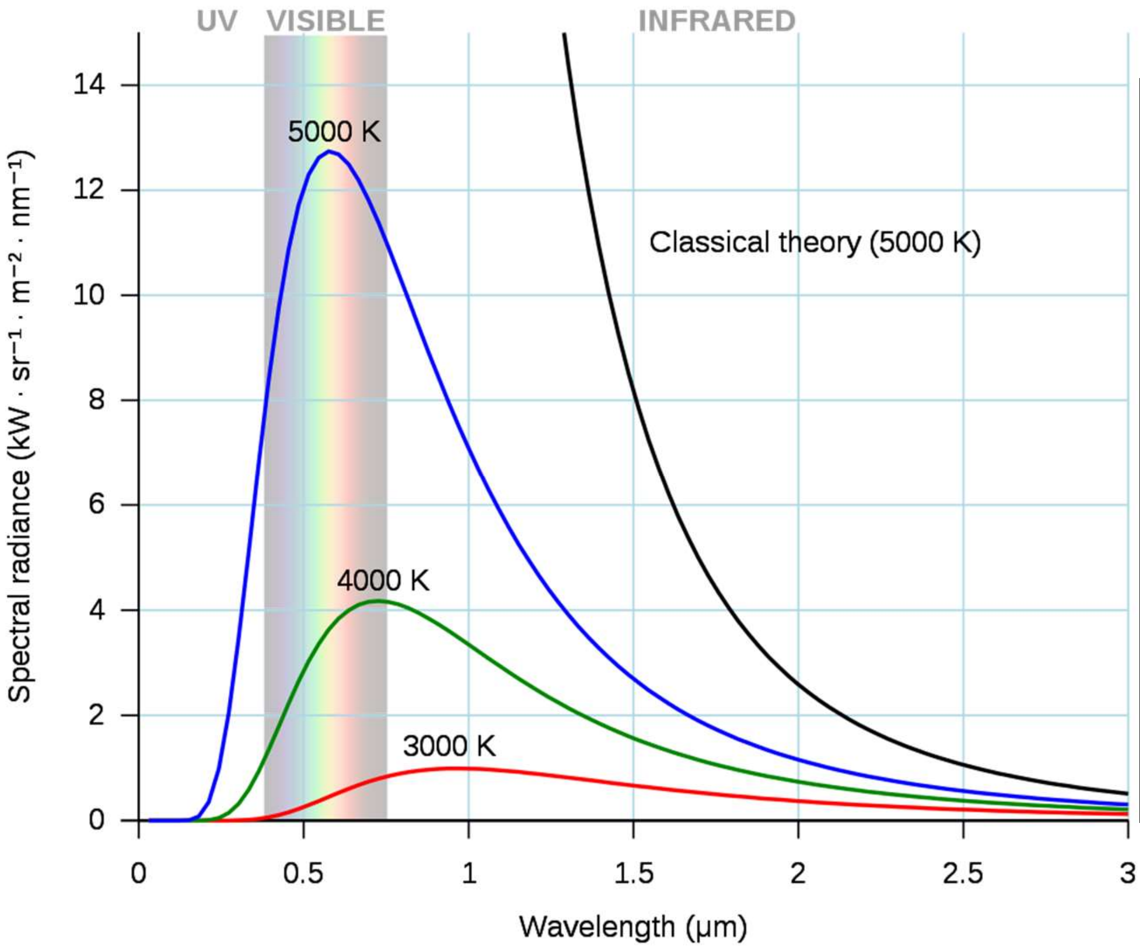
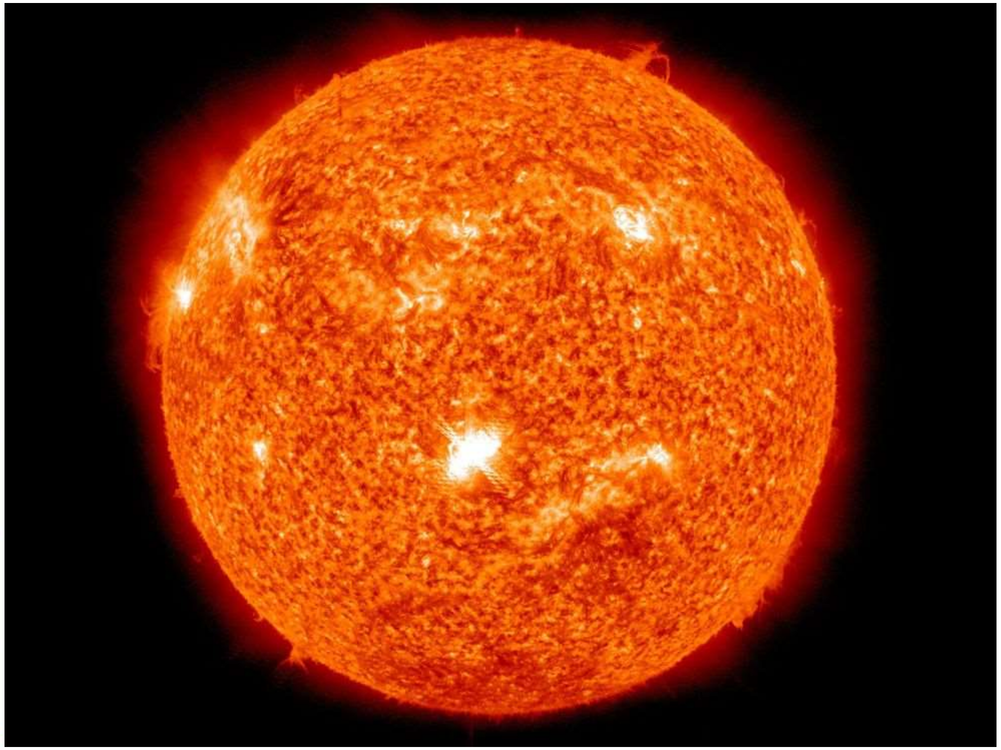
Blackbody Radiation

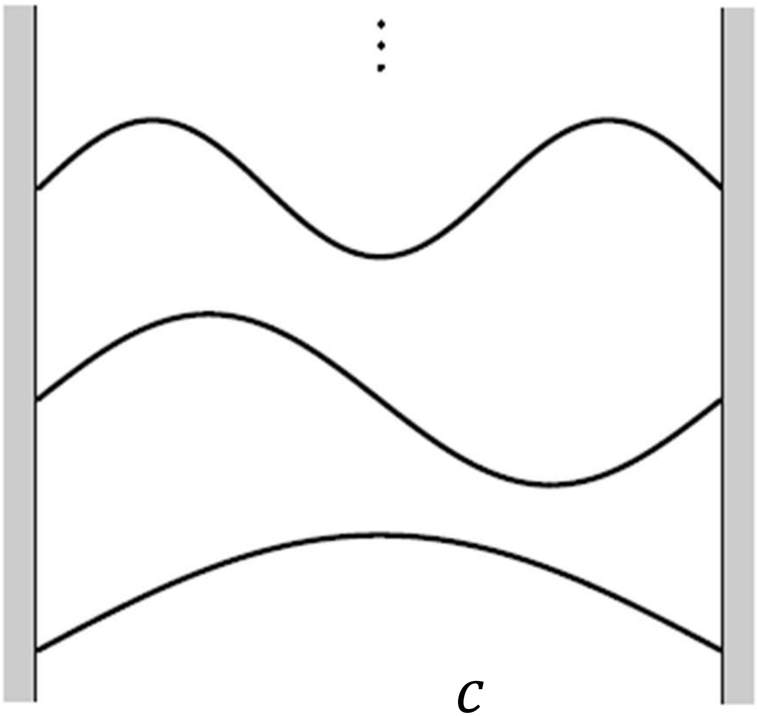
Saiqi's guest lecture
12/4/2020

Blackbodies emit electromagnetic radiation of all wavelengths.





We can analyze the electromagnetic field in a box as a superposition of standing-wave modes of various wavelengths.

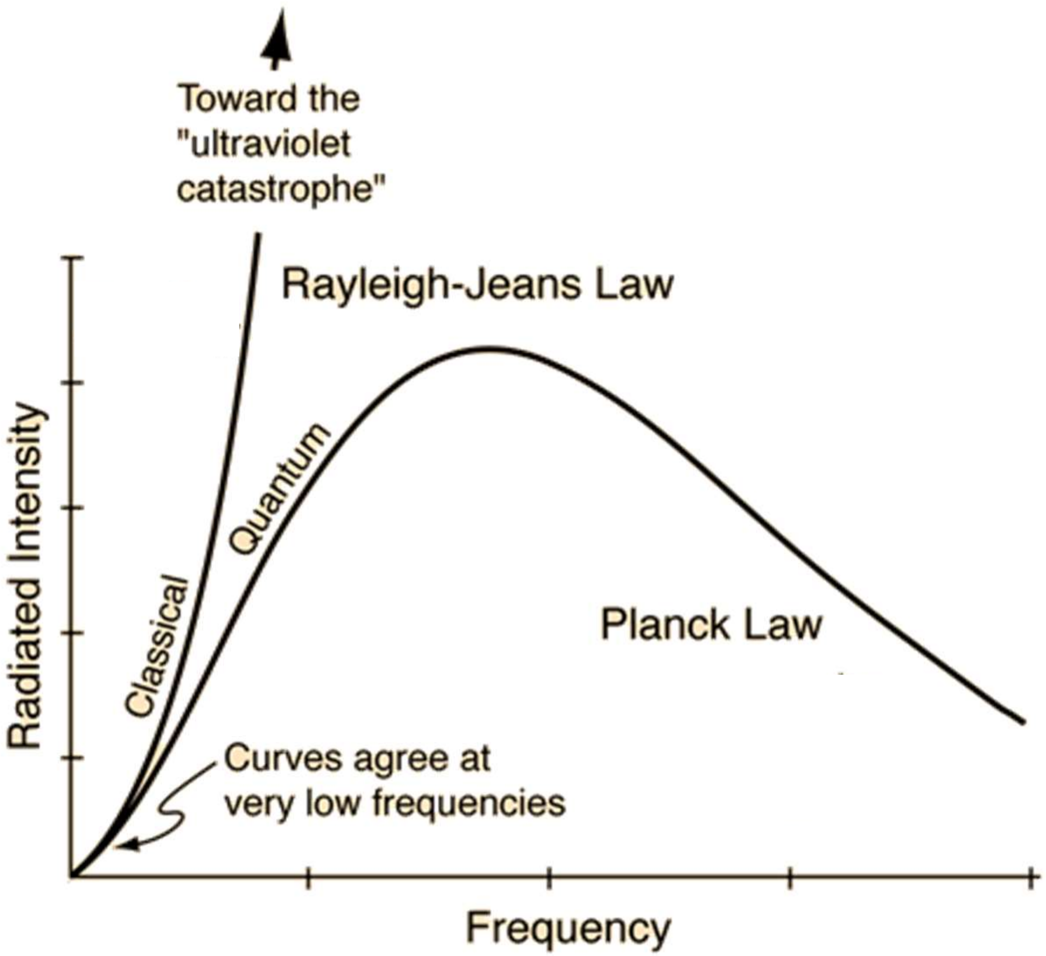


$$f = \frac{c}{\lambda}$$

Classically, each oscillator should have an average energy of kT, so the total energy would then be

$$U = kT * \infty$$

The ultraviolet catastrophe! How should we deal with it?

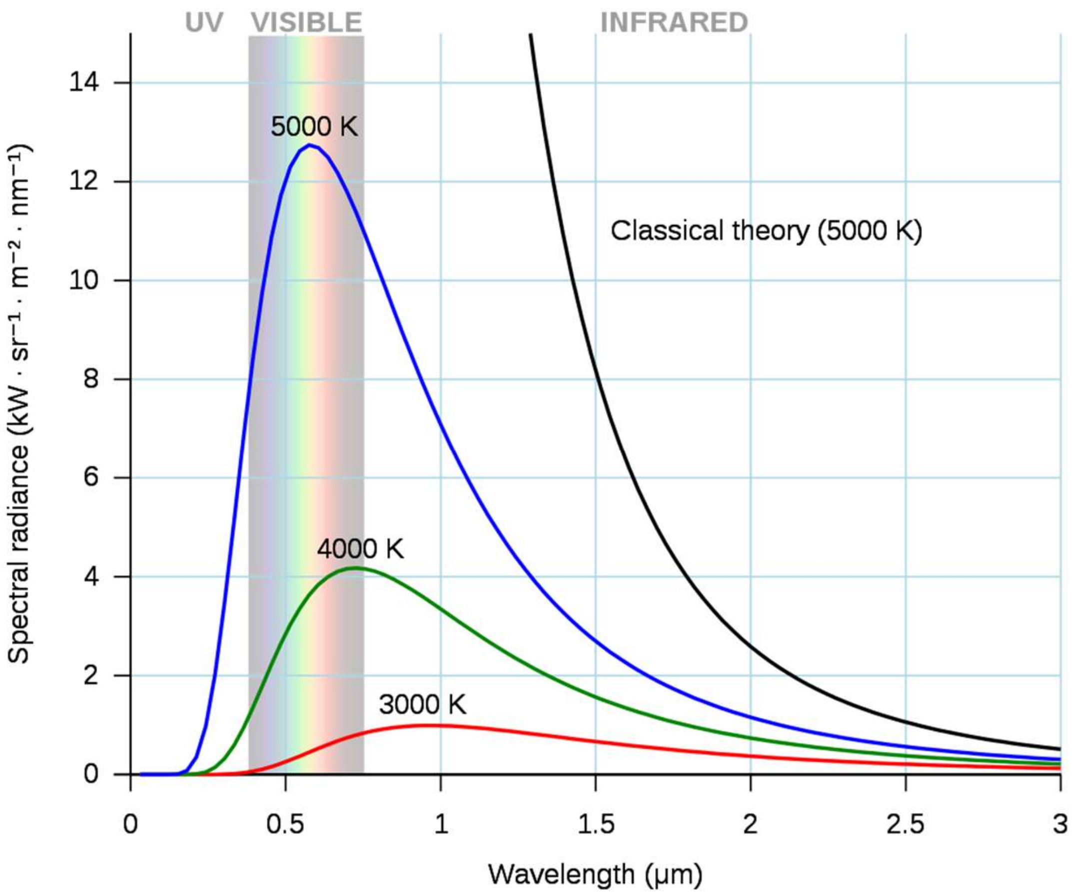


Planck's genius insight was that the energy can't take on whatever value.

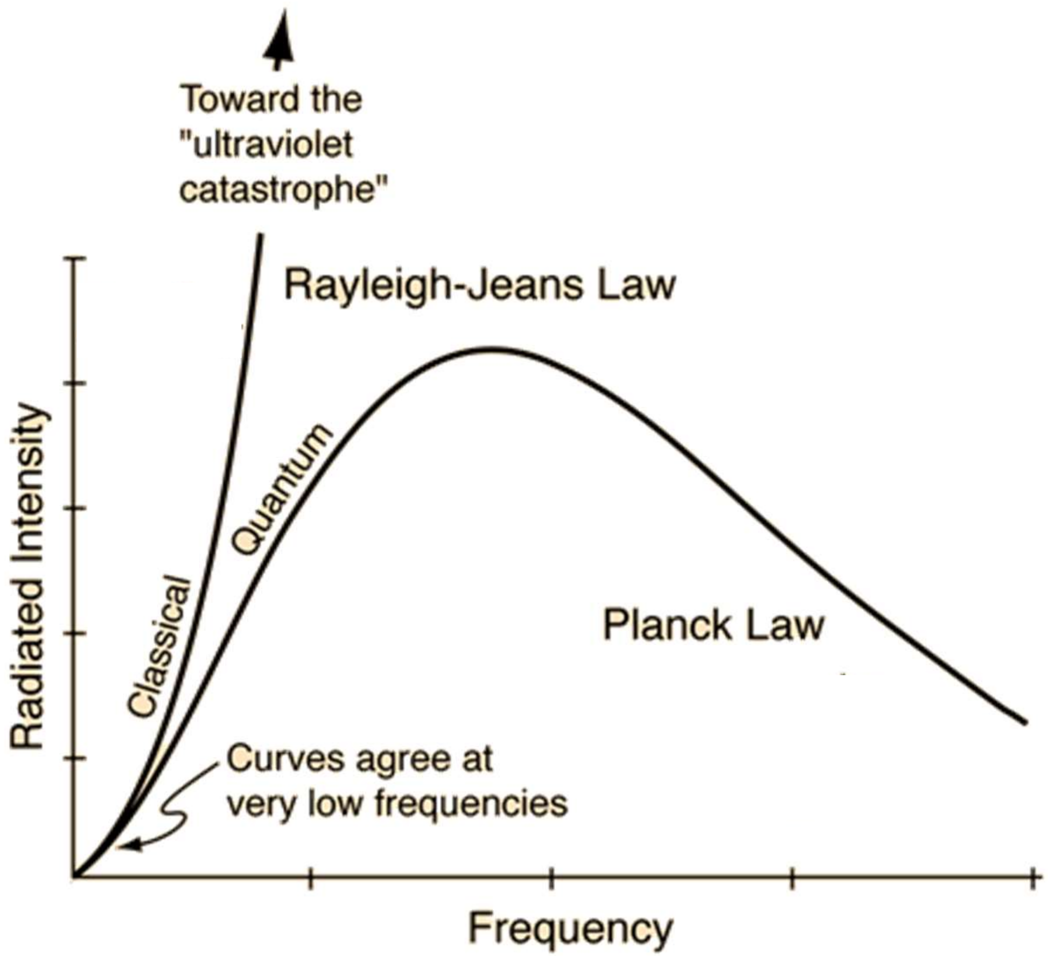
$$E_n = 0, hf, 2hf, \dots.$$

The partition function for a single oscillator is therefore

$$Z = 1 + e^{-\beta hf} + e^{-2\beta hf} + \cdots$$
$$= \frac{1}{1 - e^{-\beta hf}}$$



UV
VISIBLE
INFRARED
5000 K
Classical theory (5000 K)
4000 K
3000 K
Spectral radiance (kW · sr⁻¹ · m⁻² · nm⁻¹)
Wavelength (μm)
0
2
4
6
8
10
12
14
0.5
1
1.5
2
2.5
3



Toward the "ultraviolet catastrophe"
Rayleigh-Jeans Law
Quantum
Classical
Planck Law
Radiated Intensity
Curves agree at very low frequencies
Frequency

The average energy is $$\bar{E} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = \frac{hf}{e^{hf/kT} - 1}$$

If we think of the energy as coming in “unit” of hf, then the average number of units of energy in the oscillator is

$$\bar{n}_{P1} = \frac{1}{e^{hf/kT} - 1}$$

This formula is called the Planck distribution.

“Units” of energy in the electromagnetic field can also be thought of not only as waves, but also as particles, called photons.

They are bosons, thus satisfy the Bose-Einstein distribution:

$$\bar{n}_{\mathrm{BE}} = \frac{1}{e^{(\epsilon-\mu)/kT} - 1}$$

Compare with the previous result, we have that $\varepsilon = hf$, and $\mu = 0$ for photons.

Why this is true? Well, we can take a closer look at few examples.

Let’s consider the Helmholtz free energy:

The minimum Helmholtz free energy is attained at equilibrium with T and V held fixed.

The number of photons, N, is not constrained in a system, but to minimize F. If N then changes infinitesimally, F should be unchanged.

$$\left(\frac{\partial F}{\partial N}\right)_{T,V} = 0$$

Note that this is exactly the definition of chemical potential, which is 0.

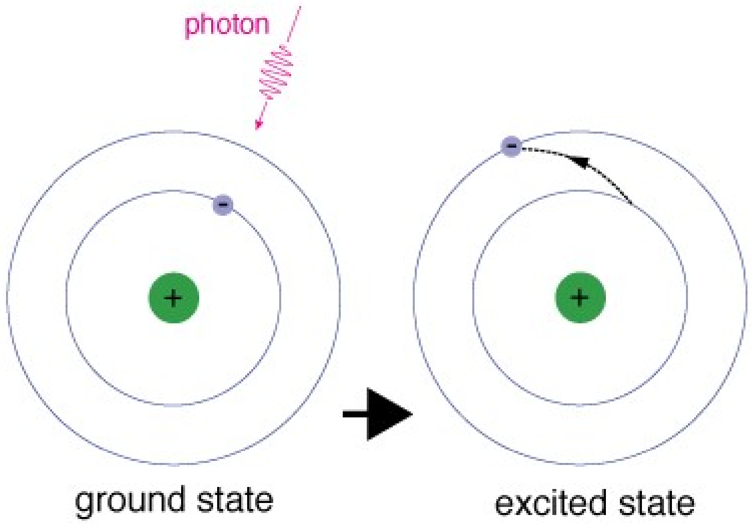
The second argument is that if we consider the process of photon (γ) being created or absorbed by an electron.

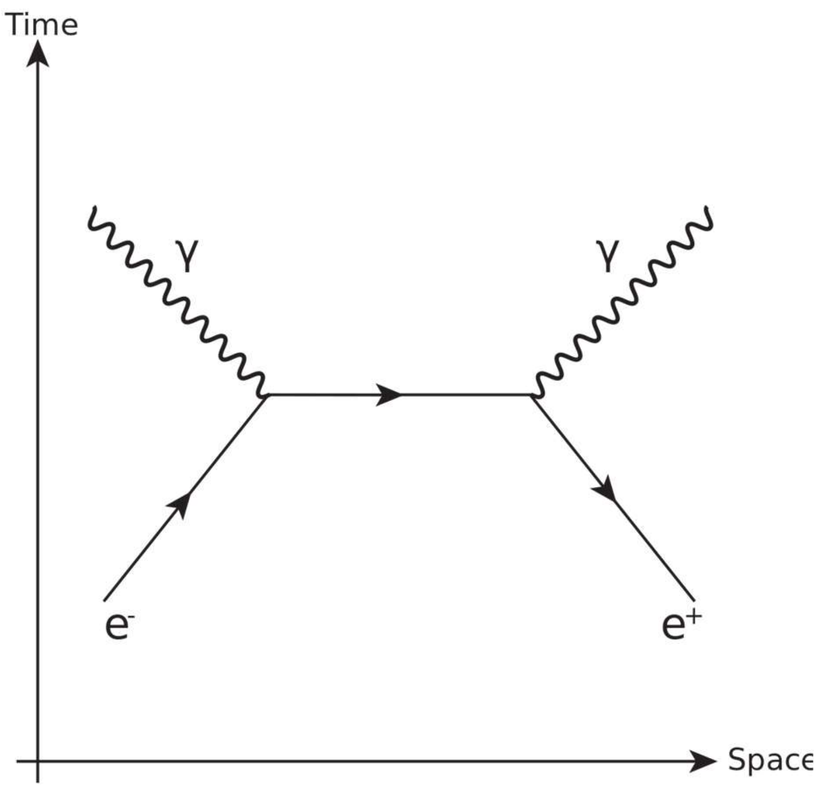
$$e \Longleftrightarrow e + \gamma$$

The equilibrium condition for such reaction to happen is that the chemical potential remains the same.

$$\mu_e = \mu_e + \mu_\gamma$$

Again, we have that the chemical potential for photons is zero.



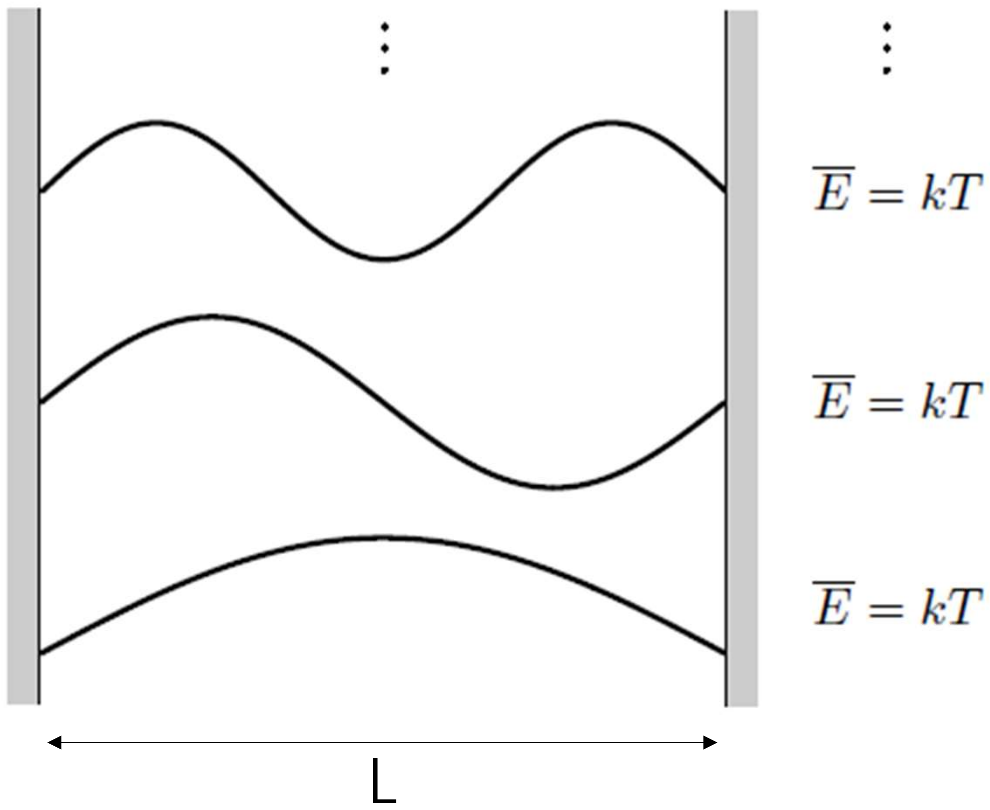


Another case would be a positron-electron collision (at low energy), and the argument is exactly the same.

I won't prove the chemical potential of a positron and a electron add up to zero here, but again in most cases, two photons are created. (The conservation laws prevented the creation of only one photon)

Again, we have that the chemical potential for photons is zero.

Now what about the total energy?



Again we consider the 1-D case, where all the waves are constrained in a box.

For a box of length L, we can easily calculate the wavelength, as well as the momentum.

$$\lambda = \frac{2L}{n}, p = \frac{h}{\lambda} = \frac{hn}{2L}.$$

Instead of our old friend $E = mc^2$, we use $E = pc = \frac{hcn}{2L}$, for photons all travel with the speed of light.

In the 3-D case, we need a sum of all the momentum vectors, and we have

$$\epsilon = c\sqrt{p_x^2 + p_y^2 + p_z^2} = \frac{hc}{2L}\sqrt{n_x^2 + n_y^2 + n_z^2} = \frac{hcn}{2L}$$

In order to get the total energy in all modes, we sum over n in all three directions, and multiply the result by 2 for each wave can hold photons with two independent polarizations. Then the total energy becomes

$$U = 2\sum_{n_x}\sum_{n_y}\sum_{n_z} \epsilon \bar{n}_{P1}(\epsilon) = \sum_{n_x, n_y, n_z} \frac{hcn}{L} \frac{1}{e^{hcn/2LkT} - 1}$$

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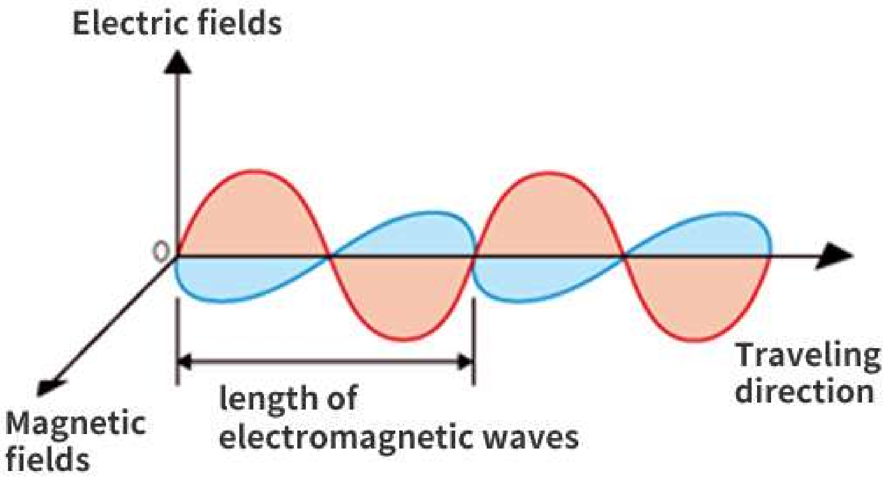

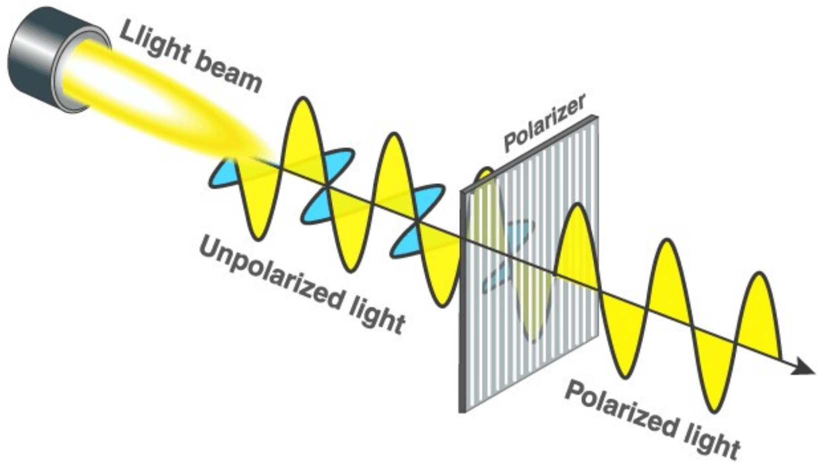

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Same as before, we convert this to an integral, but we need to be a little bit more careful with the limits.

$$U = \int_0^{\infty} \mathrm{d}n \int_0^{\pi/2} \mathrm{d}\theta \int_0^{\pi/2} \mathrm{d}\phi n^2 \sin\theta \frac{hcn}{L} \frac{1}{e^{hcn/2LkT} - 1}$$

The upper limit of n is ∞, and for θ and ϕ the upper limit is $\pi/2$.

For $\epsilon = \frac{hcn}{2L}$, $d\epsilon = \frac{hc}{2L}\, dn$, and we have:

$$U = \int_0^{\infty} dn \int_0^{\pi/2} d\theta \int_0^{\pi/2} d\phi n^2 sin\theta \frac{hcn}{L} \frac{1}{e^{hcn/2LkT} - 1}$$

$$= [sin\theta]_0^{\pi/2} [\phi]_0^{\pi/2} \int_0^{\infty} \frac{hcn^3}{L} \frac{1}{e^{hcn/2LkT} - 1} dn$$

$$= \frac{\pi}{2} \frac{(hcn)^3}{(2L)^3} \frac{8L^2}{(hc)^2} \frac{2L}{hc} \frac{1}{e^{(hcn/2L)(1/kT)} - 1} d\epsilon$$

$$= L^3 \int_0^{\infty} \frac{8\pi\epsilon^3/(hc)^3}{e^{\epsilon/kT} - 1} d\epsilon$$

But here $L^3 = V$, so $\frac{U}{V} = \int_0^{\infty} \frac{8\pi\epsilon^3/(hc)^3}{e^{\epsilon/kT}-1} d\epsilon$.

$$\frac{U}{V} = \int_0^\infty \frac{8\pi\epsilon^3/(hc)^3}{e^{\epsilon/kT} - 1} d\epsilon$$

The result if beautiful… Notice the integrand can be interpreted as energy density per unit photon energy, or the spectrum of the photons.

$$u(\epsilon) = \frac{8\pi}{(hc)^3} \frac{\epsilon^3}{e^{\epsilon/kT} - 1}$$

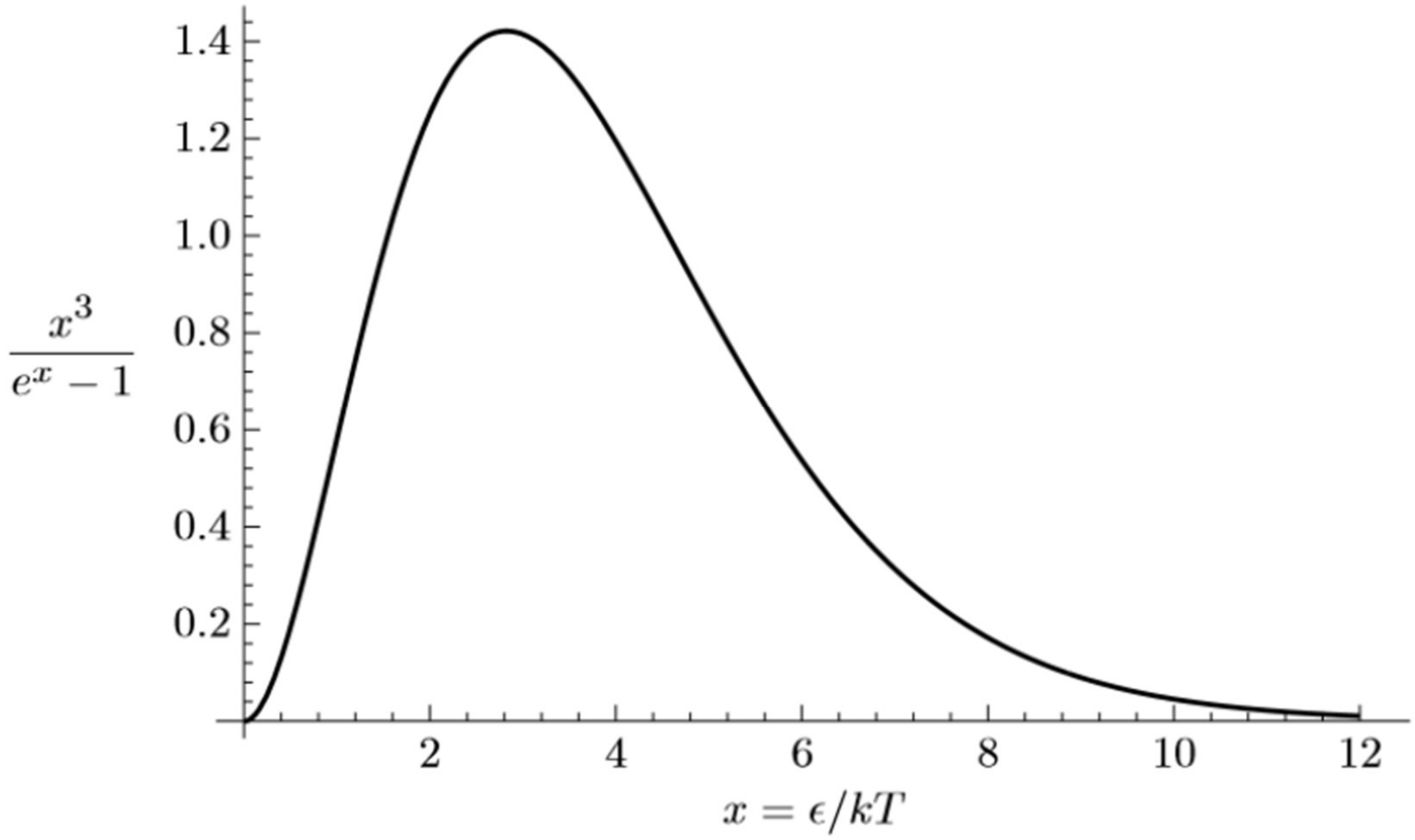
This function gives the relative intensity of the radiation as a function of photon energy. If you integrate $u(\epsilon)$ from ϵ_1 to ϵ_2, you get the energy per unit volume within that range of photon energies.

$$\frac{U}{V} = \int_0^\infty \frac{8\pi\epsilon^3/(hc)^3}{e^{\epsilon/kT} - 1} d\epsilon$$

If we change the variable, let $x = \epsilon/kT$, the equation becomes:

$$\frac{U}{V} = \frac{8\pi(\mathrm{kT})^4}{(hc)^3} \int_0^\infty \frac{x^3}{e^{\mathrm{x}} - 1} \, dx$$

Here is a nice plot of this function:



The higher temperature tend to give higher photon energies, which is known as Wien’s law.

- Thank you!