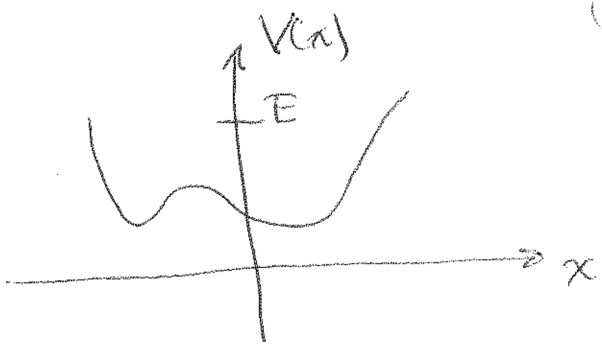
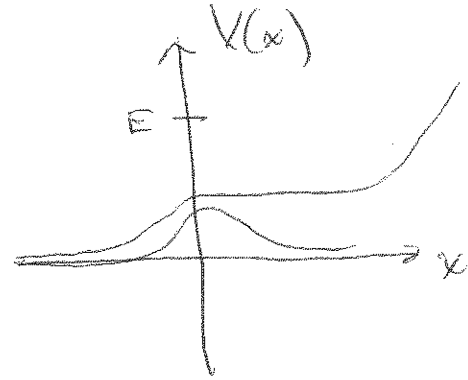
Mar 11th, 2015 Pg 1/4

Chen's

Guest Lecture

Bound & Scattering States:





$E < V(-\infty)$ and $V(\infty)$

Bound state

$E > V(-\infty)$ or $V(+\infty)$

Scattering

For ∞-sq. well or H.O.

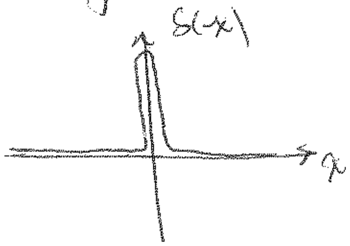
$$V(\pm\infty) \to \infty$$

always have bound states

while for the free particle

$V(x) = 0 \rightarrow$ always scattering states

Today: δ-function well. Recall



Def. $\delta(x) = \begin{cases} +\infty & x=0 \\ 0 & \text{else} \end{cases}$ with $\int_{-\infty}^{\infty} \delta(x) = 1$

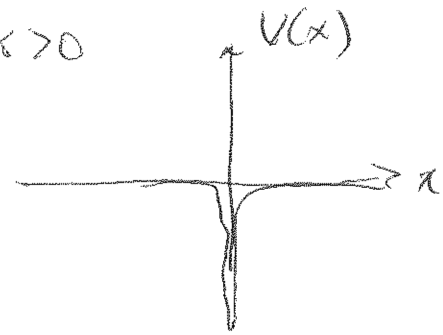
Note, $f(x)\delta(x-a) = f(a)\delta(x-a)$, so

$$\int f(x)\delta(x-a)\,dx = f(a)\int_{-\infty}^{\infty}\delta(x-a)\,dx$$

← also works for any domain around a.

$$= f(a).$$

Suppose $V(x) = -\alpha\,\delta(x)$ w/ $\alpha > 0$



The Schrodinger equation gives

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \alpha\,\delta(x)\,\psi = E\psi$$

a. Bound state $E < V(\pm\infty) = 0$.

b. Scattering state $E > 0$.

① In $x < 0$ $\delta(x) = 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \Rightarrow \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar^2}\psi$$

$$= \kappa^2\psi$$

So,

$$\psi(x) = Ae^{-\kappa x} + Be^{\kappa x} \qquad \text{with } \kappa = \frac{\sqrt{-2mE}}{\hbar}$$

② In $x > 0$ $\delta(x) = 0$ Same

$$\psi(x) = Fe^{-\kappa x} + Ge^{\kappa x} \qquad (x > 0)$$

~~Bo~~ In order for ψ not to blow up

$$A = G = 0$$

So

$$\psi(x) = B e^{\kappa x} \qquad x < 0$$

$$\psi(x) = F e^{-\kappa x} \qquad x > 0$$

There are two boundary conditions

$$\begin{cases} \psi(x) \text{ continuous} \\ \dfrac{d\psi(x)}{dx} \text{ continuous except at pts where } V = \infty. \end{cases}$$

At $x = 0$ we get

$$B e^0 = F e^0 \Rightarrow B = F.$$

So

$$\psi(x) = \begin{cases} B e^{\kappa x} & x < 0 \\ B e^{-\kappa x} & x > 0 \end{cases} = B e^{-\kappa |x|}$$

Consider,

$$-\frac{\hbar^2}{2m}\int_{-\epsilon}^{+\epsilon} \frac{d^2\psi}{dx^2}dx + \int_{-\epsilon}^{\epsilon} V(x)\psi(x)\,dx = \int_{-\epsilon}^{\epsilon} E\psi\,dx$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d\psi}{dx}\Big|_{-\epsilon}^{\epsilon} - \alpha\int_{-\epsilon}^{\epsilon} \delta(x)\psi(x)\,dx = 0$$

Then

$$\Delta\left(\frac{d\psi}{dx}\right) = \alpha\,\psi(0)\cdot\left(-\frac{2m}{\hbar^2}\right)$$

We have $\psi(0) = B$ and so

$$\Delta\left(\frac{d\psi}{dx}\right) = -\frac{2m\alpha}{\hbar}B$$

At $x \geq 0$

$$\left.\frac{d\psi}{dx}\right|_{+\epsilon} = -Bk\,e^{-kx} = -Bk$$

$$\left.\frac{d\psi}{dx}\right|_{-\epsilon} = Bk\,e^{+kx} = Bk$$

So

$$\Delta\left(\frac{d\psi}{dx}\right) = -Bk - Bk = -2Bk = -\frac{2m\alpha B}{\hbar^2}$$

$$\Rightarrow k = \frac{m\alpha}{\hbar^2} = \frac{\sqrt{-2mE}}{\hbar}$$

Then

$$E = -\frac{k^2\hbar^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}$$

To find B normalize

$$\int_{-\infty}^{\infty}|\psi|^2\,dx = B^2\left(\int_{-\infty}^{0} e^{2kx} + \int_0^{\infty} e^{-2kx}\,dx\right)$$

$$= 2B^2\int_0^{\infty} e^{-2kx}\,dx = 2B^2\left(-\frac{e^{-2kx}}{2k}\right)\Bigg|_0^{\infty} = \frac{B^2}{k} = 1 \Rightarrow B = \sqrt{k}$$

So

$$\boxed{E = -\frac{m\alpha^2}{2\hbar^2}}$$

$$\boxed{\psi(x) = \frac{\sqrt{m\alpha}}{\hbar}\,e^{-\frac{m\alpha}{\hbar^2}|x|}}$$

- Guess
- Progression was very clear.
- Very clear and left important things on board
- Easy to take notes