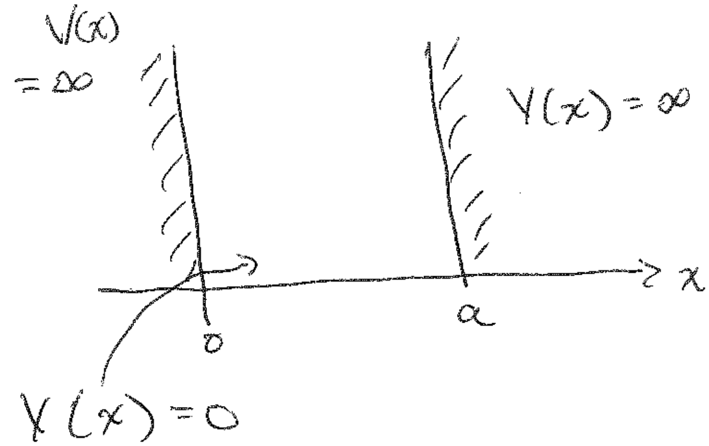
Quantum Mechanics

Loren guest lecture

The infinite square well:

Say we have a particle in a square well



Outside the well $V = \infty$ so if

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

is to have a finite solution

$$\psi(x) = 0 \qquad \begin{array}{l} -\infty < x < 0 \\ a < x < \infty \end{array}$$

$$\psi(x) \neq 0 \qquad 0 < x < a$$

Inside the well

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \implies \frac{d^2\psi}{dx^2} = -\frac{2mE}{\hbar}\psi$$

Define Wave number,

$$k = \frac{\sqrt{2mE}}{\hbar}$$

so that

$$\frac{d^2\psi}{dx^2} = -k\psi$$

and this ODE is the same as that of a simple harmonic oscillator,

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

To find A and B let's look at boundary conditions

$$\psi(0) = \psi(a) = 0$$

for ψ to be continuous at the boundary. Then at $x=0$

$$\psi(0) = B\cos(0) = B = 0$$

$$\Rightarrow B=0$$

and at $x=a$

$$\psi(a) = A\sin(ka) = 0$$

then

$$ka = 0, \pi, 2\pi, 3\pi, \ldots$$

This implies

interesting $\downarrow$ values

$$k_n = \frac{n\pi}{a} \quad n = 1, 2, 3$$

and so

$$\psi(x) = A \sin\left(\frac{n\pi}{a} x\right)$$

We also had

$$k = \frac{\sqrt{2mE}}{\hbar}$$

so putting these together

$$\frac{\sqrt{2mE}}{\hbar} = \frac{n\pi}{a}$$

$$\Rightarrow \boxed{E = \frac{n^2\pi^2\hbar^2}{2m\,a^2} \quad n = 1, 2, \ldots}$$

Energy in Quantum Mechanics is quantized because of boundary conditions.

Let's normalize

$$\psi(x) = A \sin\left(\frac{n\pi}{a} x\right)$$

$$\Rightarrow \int_0^a A^2 \sin^2\left(\frac{n\pi}{a} x\right) dx = A^2 \frac{a}{2} = 1$$

or $A = \sqrt{\frac{2}{a}} \Rightarrow \boxed{\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)}$