Michael's guest Apr 1st, 2015 P1/4

Lecture on

Spin Angular Momentum

i. Spin

ii. Spin 1/2

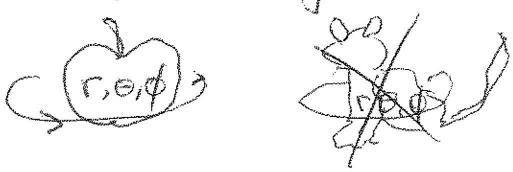
i-Spin

Classically, orbital system like Earth around sun,

$\vec{L} = \vec{r} \times \vec{p}$ — motion of center of mass

$\vec{S} = I\vec{\omega}$ — motion about C.O.M

Quantum mechanically this can be thought of as electron around nucleus. The idea of orbital angular momentum holds.



Spin is intrinsic to ~~the~~ elementary particle not dependent on position variables.

Algebraically, we borrow from orbital angular momentum

$$[S_x, S_y] = i\hbar S_z \quad \text{and} \quad ?$$

Also, $S^2|s,m\rangle = \hbar^2 s(s+1)|s,m\rangle$, $S_z|s,m\rangle = \hbar m|s,m\rangle$.

$S_{\pm} \equiv S_x \pm i S_y$

ii - Spin-1/2: Spin 1/2 particles make up all of ordinary matter: electrons, protons, neutrons, quarks, photons.

Just two eigenstates:

$|s\ m\rangle$

$|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$ $\uparrow$ (spin up)

$|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$ $\downarrow$ (spin down)

We can also express these as basis vectors of a general state:

$$\chi = \begin{pmatrix} a \\ b \end{pmatrix} = a\chi_+ + b\chi_-$$

where

$$\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We call these "spinors".

$$S^2 \chi_+ = \hbar^2 s(s+1)\chi_+$$

$$= \hbar^2 \tfrac{1}{2}\left(\tfrac{1}{2}+1\right)\chi_+ = \tfrac{3}{4}\hbar^2 \chi_+$$

$$S_z \chi_+ = m\hbar \chi_+ = \tfrac{1}{2}\hbar \chi_+$$

Similarly

$$S_z \chi_- = -\tfrac{1}{2}\hbar \chi_-$$

While

$$S_+ \chi_- = \hbar \sqrt{s(s+1) - m(m\pm 1)}\; |s, (m\pm 1)\rangle$$

$$= \hbar \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1) - (-\tfrac{1}{2})(-\tfrac{1}{2}+1)}\; |\tfrac{1}{2}\ \tfrac{1}{2}\rangle$$

$$= \hbar\ |\tfrac{1}{2}\ \tfrac{1}{2}\rangle.$$

and again $S_- \chi_+ = \hbar \chi_-$

We also have

$$S_+ \chi_+ = S_- \chi_- = 0$$

Recall, $S_\pm = S_x \pm i S_y \Rightarrow S_x = \frac{1}{2}(S_+ + S_-)$

and similarly $S_y = \frac{1}{2i}(S_+ - S_-)$. Now we can compute

$$S_x \chi_+ = \tfrac{1}{2}(S_+ + S_-)\chi_+ = 0 + \tfrac{1}{2} S_- \chi_+$$

$$= \tfrac{\hbar}{2} \chi_-$$

Again

$$S_y \chi_+ = -\frac{\hbar}{2i}\chi_- \quad , \quad S_y \chi_- = \frac{\hbar}{2i}\chi_+$$

Now, let's deduce the matrix form of the spin angular momentum operators:

$$S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then

$$S^2 \chi_+ = \frac{3}{4}\hbar^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{3}{4}\hbar^2 \begin{pmatrix} a \\ c \end{pmatrix} = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad a = 1 \quad \text{and} \quad c = 0$$

while

$$S^2 \chi_- = \frac{3}{4}\hbar^2 \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

from before

$$= \frac{3}{4}\hbar^2 \begin{pmatrix} b \\ d \end{pmatrix} \overset{d}{=} \frac{3}{4}\hbar^2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad b = 0 \quad d = 1$$

Finally then, $S^2 = \frac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.