# Quantum Mechanics — Feb $16^{th}$, 2015

## Lect 10

I Last time

II Observable determinate States

(or eigenstates of an hermitian operator)

III Applications: t-independent Schrödinger Eq and stationary states

hermitian: iff $\hat{Q}^{\dagger} = Q$.

- Studied determinate states

$$\sigma^2 = 0$$

$$\Rightarrow \langle (\hat{Q}-q)\Psi \mid (\hat{Q}-q)\Psi \rangle = 0$$

But, then

$$\boxed{\hat{Q}\,\Psi = q\,\Psi}$$ Eigenvalue equation

$\Psi$ is an eigenstate of $\hat{Q}$

I We argued that

$$\boxed{\text{Observable s are represented by Hermitian operators}}$$

- Useful characterization

Adjoint: $\langle f \mid \hat{Q} g \rangle \overset{d}{=} \langle \hat{Q}^{\dagger} f \mid g \rangle$ always true

$q$ is an eigenvalue of $\hat{Q}$

$$\boxed{\text{Determinate states are eigenfunctions of } \hat{Q}}$$

II Let's put these two ideas together

Two important cases $\begin{cases} \text{discrete set of } \{q_i\}_{i=1}^{N} \text{ (called the spectrum of } \hat{Q}) \\ \text{"discrete spectrum"} \\ \text{continuous spectrum} \end{cases}$

<u>discrete spectrum</u>: Two theorems about hermitian operators

<u>Thm 1</u>: The e-values are real

<u>Pf</u>: suppose $\hat{Q} f = q f$

and $$\langle \hat{Q} f | f \rangle = \langle f | \hat{Q} f \rangle$$

Then,
$$\langle q f | f \rangle = \langle f | q f \rangle$$

and
$$q^* \langle f | f \rangle = q \langle f | f \rangle$$

But $\langle f | f \rangle \neq 0$ and so

$$q = q^* \Rightarrow \boxed{q \in \mathbb{R}}$$

[Aside: suppose $q = x + iy$ then

$q = q^* \Rightarrow x + iy = x - iy$

$\Rightarrow x = x$ and $y = -y \Rightarrow y = 0$

$\Rightarrow q = x$ is real. ]

<u>Thm 2</u>: E-functions for distinct $q$ are orthogonal

<u>Pf</u>: suppose
$$\hat{Q} f = q f \text{ and } \hat{Q} g = q' g$$

Then
$$\langle f | \hat{Q} g \rangle = \langle \hat{Q} f | g \rangle$$

$$\Rightarrow \quad q' \langle f | g \rangle = q^* \langle f | g \rangle$$

But by assumption $q$ and $q'$ are distinct, so, $q \neq q' \Rightarrow$

$$\boxed{\langle f | g \rangle = 0}$$

Note that this theorem tells us nothing about distinct $f$ and $g$ having equal $q$. But, the Gram-Schmidt procedure allows you to <u>construct</u> an orthonormal basis amongst all e-functions with equal e-values.

<u>Thm 3</u>: If ~~#~~ $\dim \mathcal{H}$ = finite # then the eigenvectors of any hermitian $\hat{Q}$ span all of $\mathcal{H}$.

There is no such thm. in the infinite dimensional context — but, this is required for quantum theory.

Nonetheless we can proceed:

<u>Ex.</u> Find the e-functions and e-values of $\hat{p}$, call 'em $f_p$.

$$\frac{\hbar}{i}\frac{\partial}{\partial x} f_p(x) = p\, f_p(x)$$

e-value of $\hat{p}$

General solution

$$f_p(x) = A\, e^{ipx/\hbar}$$

– not normalizable and hence doesn't live in Hilbert Space.

So along with Dirac we take it as an axiom:

The e-functions of an observable operator are <u>complete</u>.

<u>Continuous spectrum:</u>

The e-functions in this case are not normalizable!

Nonetheless,

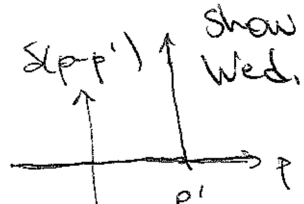
$$\int_{-\infty}^{\infty} f_{p'}^*(x)\, f_p(x)\, dx$$

$$= |A|^2 \int_{-\infty}^{\infty} e^{i(p-p')x/\hbar}$$

We'll show Wed.

$$= |A|^2\, 2\pi\hbar\, \delta(p-p')$$

Dirac delta function




$$\delta(p-p') = \begin{cases} \infty & p = p' \\ 0 & p \neq p' \end{cases} \quad \text{or} \quad \delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0. \end{cases}$$

and such that $\int \delta(x)\, dx = 1$

Then if we take $A = \frac{1}{\sqrt{2\pi\hbar}}$

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

and

$$\langle f_{p'} | f_p \rangle = \delta(p-p').$$

This is ~~an~~ awfully similar to ~~orthogonal~~ orthonormality — it's a sort of continuum version. — Dirac orthonormality.

Notice the pleasant convergence of our ideas

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

is a plane wave with

$$\boxed{\lambda = \frac{2\pi\hbar}{p} = \frac{h}{p}}$$

We have a new appreciation for these subtle eigenfunctions ⇝ they're not in the Hilbert Space.



Even better the $f_p$ are complete:

$$f(x) = \int_{-\infty}^{\infty} c(p) f_p(x)\, dp$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p) e^{ipx/\hbar} dp$$

and

$$\langle f_{p'} | f \rangle = c(p')$$

show tomorrow.

If the spectrum of a hermitian $\hat{Q}$ is continuous the e-functions are not normalizable, and don't live in $\mathcal{H}$. Nonetheless they are Dirac orthonormalizable and complete.

III ~~Let's return to~~

$$\cancel{i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + V\Psi}.$$