Quantum Mechanics — Feb 20th, 2015 — P1/4

Lect 12

I Last time

II Why the well?

III Quantum state of square well

IV The Harmonic Oscillator

I • Illustrated separation of variables on the Schrödinger PDE.

• Arrived at

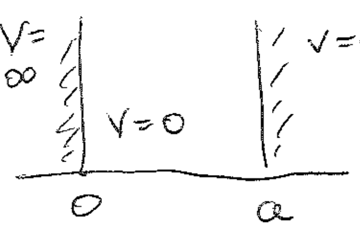
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi$$

or $\quad \hat{H}\psi = E\psi.$

and $\quad \varphi = e^{-iEt/\hbar}$

with $\quad \Psi = \varphi\psi.$

• Loren derived



$$E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad n = 1, 2, \ldots$$

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

II Why the well?

It's a nice model because it's exactly solvable.

But it's also useful in modeling physics, e.g. Nuclear shell model. Protons and neutrons are tightly bound in the nucleus due to strong force — the potential can be seen as a <u>finite</u> square well. We'll study this soon. Also used in optoelectronics, quantum well laser, quantum well infrared detector etc.

III Don't forget that the physical state is

$$\Psi(x,t) = \psi(x)\varphi(t)$$

So we put our findings together

Linear combinations of stationary states give a general solution of the Schrödinger PDE when $V = V(x)$.

<u>Strategy</u>: Tell me $V(x)$ and $\Psi(x,0)$ ← initial condition

then

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$ ← completeness of e-functions

and then indeed

$$\Psi_n(x,t) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{-i\frac{n^2\pi^2\hbar}{2ma^2}t}$$

and a general state is

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)e^{-i\frac{n^2\pi^2\hbar}{2ma^2}t}$$

Let's return to our question: why care about stationary states? Reason 3:

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n e^{-iE_n t/\hbar}$$

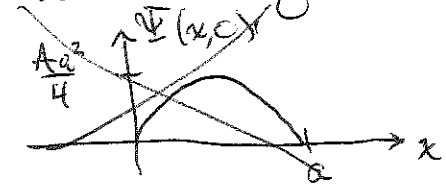
is a solution of Sch.'s eqn.

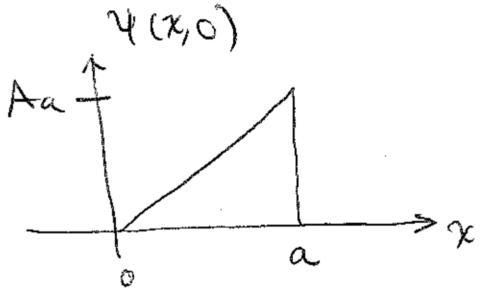
Check it!

Let's do an example:

$\Psi(x,0) = A x$ ~~$(a-x)$~~ $(0 \le x \le a)$

~~in the ∞ square well.~~





Normalize:

$$\int_0^a A^2 x^2 \, dx = 1$$

$$\Rightarrow A^2 \frac{a^3}{3} = 1 \Rightarrow A = \sqrt{\frac{3}{a^3}}$$

Find C_n:

$$C_n = \sqrt{\frac{3}{a^3}} \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) x \, dx$$

tells you the probability of measuring E_n, as such

$$\sum_{n=1}^{\infty} |C_n|^2 = 1$$

Let's check our case

$$|C_n|^2 = \frac{(-1)^{2n+2} 6}{n^2 \pi^2} = \frac{6}{n^2 \pi^2}$$

$$\Rightarrow C_n = \frac{\sqrt{6}}{a^2} \left. \frac{-a^2 \cos\left(\frac{n\pi x}{a}\right)}{n^2 \pi^2} \right|_0^a$$

$$= \frac{-\sqrt{6} \cos(n\pi)}{n^2 \pi^2} = \frac{(-1)^{n+1}\sqrt{6}}{n^2 \pi^2}$$

So,

$$\boxed{\Psi(x,t) = \sqrt{\frac{2}{a}} \frac{\sqrt{6}}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \sin\left(\frac{n\pi}{a}x\right) \times e^{-i\frac{n^2\pi^2\hbar}{2ma^2}t}}$$

times

We have that

$$|C_n|^2$$

$$\sum_{n=1}^{\infty} \frac{6}{n^2 \pi^2} = 1 \checkmark$$

We also have

$$\langle Q \rangle = \sum_n P_n Q_n$$

So

$$\boxed{\langle \hat{H} \rangle = \sum_{n=1}^{\infty} |C_n|^2 E_n}$$

IV We have

$$F = -kx = m\frac{d^2x}{dt}$$

with soln

$$x(t) = A\sin(\omega t) + B\cos(\omega t)$$

with $\omega = \sqrt{\frac{k}{m}}$,

Viewed as a potential problem

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$$

$$V(x) = \underset{\uparrow \text{ just a shift}}{V(x_0)} + \cancel{V'(x_0)}^{\,0}(x-x_0) + \frac{1}{2}V''(x_0)(x-x_0)^2 + \cdots$$

So

$$V(x) \approx \frac{1}{2}V''(x_0)(x-x_0)^2$$

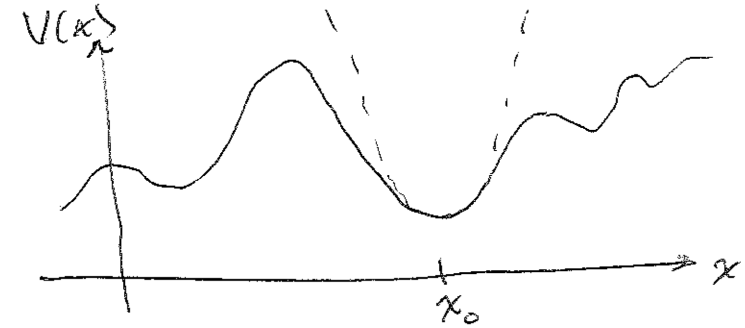
Then we tackle

makes this ODE tricky

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi$$

We'll use the algebraic "ladder operator" method of solution for

The reason this potential is so important is that it resembles any potential near its minimum



now,