# Quantum Mechanics
Lect. 13

Feb 23th, 2015 P1/4

I Last time

II The harmonic oscillator: algebraic method

So

$$C_n = \sum \int_{-\infty}^{\infty} \psi_n^* \, \Psi(x,0)\, dx$$

and then

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n e^{-\frac{iE_n t}{\hbar}}$$

Applied this to an example in the square well.

• Pointed out the great generality

I

• Square wells, although quite ideal, are useful in applications

• Found a general state from its initial condition

$$\Psi(x,0) = \sum_{n=1}^{\infty} C_n \psi_n(x)$$

of harmonic potentials.

Near a minimum:

$$V(x) \approx \frac{1}{2} V''(x_0)\,(x-x_0)^2$$

↑ position of min.

II So we turn to

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2} m\omega^2 x^2 \psi = E\psi$$

That is, to

$$V(x) = \frac{1}{2} m\omega^2 x^2 = \frac{1}{2} k x^2, \quad \omega = \sqrt{\frac{k}{m}}$$

Notice that because the independent variable $x$ shows up in this $2^{nd}$ order ODE, it does not fall into one of the standard tool boxes you probably know (such as guessing $\psi(x) = e^{-\lambda x}$)

We'll learn a general treatment later, but for now we'll use a special algebraic method of "ladder operators". It is <u>special</u> but an essential framework that does recur in your study of Q.M.

<u>Algebraic Method</u>: First write

$$\frac{1}{2m}\left[\hat{p}^2 + (m\omega\hat{x})^2\right]\psi = E\psi.$$

Notice that we can factor $\hat{H}$(!)

For complex #s:

$$u^2 + v^2 = (iu+v)(-iu+v)$$

but we have the added complication that $\hat{p}$ and $\hat{x}$ are operators.

Let's try anyway ← purely to make things nice

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega\hat{x})$$

Then $$\hat{a}_-\hat{a}_+ = \frac{1}{2\hbar m\omega}(i\hat{p} + m\omega\hat{x})(-i\hat{p} + m\omega\hat{x})$$

$$= \frac{1}{2\hbar m\omega}\left[\hat{p}^2 + (m\omega\hat{x})^2 - im\omega(\hat{x}\hat{p} - \hat{p}\hat{x})\right]$$

For operators (and matrices) order matters!

We call

$$[A,B] = AB - BA$$

the commutator of $A$ and $B$.

So $$\hat{a}_-\hat{a}_+ = \frac{1}{2\hbar m\omega}\left[\hat{p}^2 + (m\omega\hat{x})^2\right] - \frac{i}{2\hbar}[\hat{x},\hat{p}]$$

**Warning!** ~~D~~Don't treat operators ~~as~~ like ~~it's~~ it's too easy to make mistakes.

To do it carefully always act on a test function, say $f(x)$,

$$[\hat{x},\hat{p}]\,f(x) = (\hat{x}\hat{p}-\hat{p}\hat{x})\,f(x)$$

$$= \hat{x}\,\frac{\hbar}{i}\frac{df}{dx} - \hat{p}\;x\,f(x)$$

$$= x\,\frac{\hbar}{i}\frac{df}{dx} - \frac{\hbar}{i}\frac{d}{dx}(xf)$$

$$= -\frac{\hbar}{i}\,f = i\hbar f$$

So,

$$\boxed{[\hat{x},\hat{p}] = i\hbar}$$

known as the "canonical commutation relation"

Returning to $\hat{H}$ we have

$$\hat{a}_-\hat{a}_+ = \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}$$

or

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right)$$

Order matters – if we'd computed

$$\hat{a}_+\hat{a}_- = \frac{1}{\hbar\omega}H - \frac{1}{2},$$

that is,

$$[\hat{a}_-,\hat{a}_+] = 1$$

and

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right).$$

The Schrödinger eqn. becomes

$$\hbar\omega\left(\hat{a}_\pm\hat{a}_\mp \pm \frac{1}{2}\right)\psi = E\psi$$

<u>The prestige</u>: Here's the magic. Suppose $\psi$ is a solution w/ energy $E$ then $\hat{a}_+\psi$ is a new solution w/ energy $(E+\hbar\omega)$, that is,

$$\hat{H}(\hat{a}_+\psi) = (E+\hbar\omega)(\hat{a}_+\psi).$$

Pf: $\hat{H}(\hat{a}_+\psi) = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)(\hat{a}_+\psi)$

$= \hbar\omega\left(\hat{a}_+\hat{a}_-\hat{a}_+ + \frac{1}{2}\hat{a}_+\right)\psi$

$= \hbar\omega\,\hat{a}_+\left(\hat{a}_-\hat{a}_+ + \frac{1}{2}\right)\psi$

$[\hat{a}_-,\hat{a}_+]=+1$

$= \hbar\omega\,\hat{a}_+\left(\hat{a}_+\hat{a}_- + 1 + \frac{1}{2}\right)\psi$

$= \hat{a}_+\left(\hat{H} + \hbar\omega\right)\psi$

$= \hat{a}_+(E+\hbar\omega)\psi$

$= (E+\hbar\omega)\,\hat{a}_+\psi$

Is the ladder infinite? No!

On Wed. you will prove that

$E \geq V_{min}$.

So, there must be a state, call it $\psi_0$, such that

$\hat{a}_-\psi_0 = 0$

Let's try.

You won't be surprised that

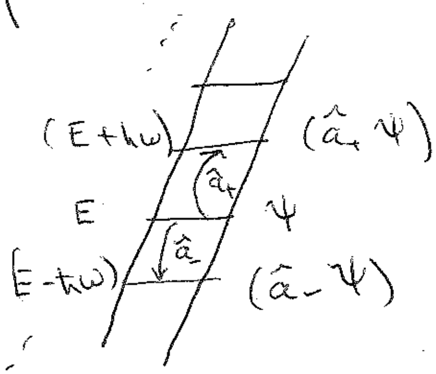
$\hat{H}(\hat{a}_-\psi) = (E-\hbar\omega)(\hat{a}_-\psi)$.

Remarkable tool

The a's are ladder operators

$\hat{a}_+$ "raising op"

$\hat{a}_-$ "lowering op"




$$\frac{1}{\sqrt{2\hbar m\omega}}\left(i\frac{\hbar}{i}\frac{d}{dx} + m\omega x\right)\psi_0 = 0$$

$$\Rightarrow \hbar\frac{d\psi_0}{dx} = -m\omega x\,\psi_0$$

or

$$\ln\psi_0 = -\frac{1}{2}\frac{m\omega}{\hbar}x^2 + \text{const}$$

$$\Rightarrow \psi_0 = A\,e^{-\frac{m\omega x^2}{2\hbar}}$$

A Gaussian!

Normalize:

$$\int_{-\infty}^{\infty}|A|^2 e^{-\frac{m\omega x^2}{\hbar}}\,dx = |A|^2\sqrt{\frac{\pi\hbar}{m\omega}} \Rightarrow A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$