I Last time

II All the oscillator States from the ladder

Using $\hat{a}_- \psi_0 = 0$

we found

$$\psi_0 = A\, e^{-\frac{m\omega x^2}{2\hbar}}$$

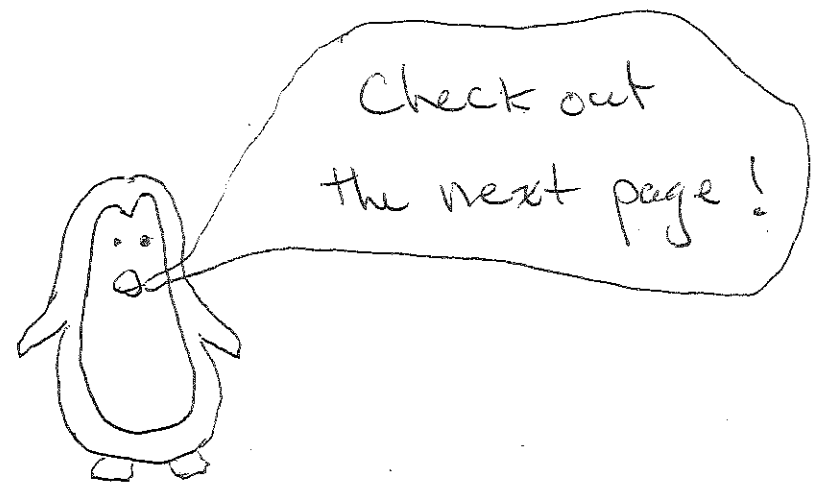
a gaussian! Normalizing

$$\int_{-\infty}^{\infty} |A|^2 e^{-\frac{m\omega x^2}{\hbar}}\, dx = |A|^2 \sqrt{\frac{\pi\hbar}{m\omega}}$$

$$\Rightarrow A = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$$

I To summarize last time let's try to fill in the table 1 (see last page)

- We were in the midst of finding the "ground state" ψ_0



II All the oscillator states from the ladde

The ground state of the harmonic oscillator is

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}}$$

It's energy is

$$\hbar\omega\left(\hat{a}_+\hat{a}_- + \tfrac{1}{2}\right)\psi_0 = E_0\psi_0$$

but $\hat{a}_-\psi_0 = 0$, so

$$\tfrac{1}{2}\hbar\omega\,\psi_0 = E_0\psi_0 \Rightarrow \boxed{E_0 = \frac{\hbar\omega}{2}}$$

$$\hat{a}_+\psi_n = c_n\psi_{n+1} \qquad \hat{a}_-\psi_n = d_n\psi_{n-1}$$

← some constants

What are c_n and d_n? Note

$$\langle f | \hat{a}_\pm g\rangle = \langle \hat{a}_\mp f | g\rangle,$$

that is, $\hat{a}_+^\dagger = \hat{a}_-$ and $\hat{a}_-^\dagger = \hat{a}_+$

Pf: $\langle f|\hat{a}_\pm g\rangle = \frac{1}{\sqrt{2\hbar m\omega}} \int_{-\infty}^{\infty} f^*\left(\mp i\hat{p} + m\omega\hat{x}\right) g\, dx$

But you've shown $\hat{p}^\dagger = \hat{p}$ and $\hat{x}^\dagger = \hat{x}$, so only the i factor matter

$$= \frac{1}{\sqrt{2\hbar m\omega}} \int_{-\infty}^{\infty} \left((\pm i\hat{p} + m\omega\hat{x})f\right)^* g\, dx$$

$$= \langle \hat{a}_\mp f | g\rangle. \blacksquare$$

Then

normalization const.

$$\boxed{\psi_n(x) = A_n\left(\hat{a}_+\right)^n\psi_0(x)}$$

and

$$\boxed{E_n = \left(n + \tfrac{1}{2}\right)\hbar\omega}$$

gives all the harmonic oscillator solutions!

In fact, we can even find A_n algebraically. Well,

Put this together with

$$\langle \hat{a}_\pm\psi_n | \hat{a}_\pm\psi_n\rangle = \langle \hat{a}_\mp\hat{a}_\pm\psi_n | \psi_n\rangle$$

and $\hat{a}_+\hat{a}_-\psi_n = n\psi_n$, $\hat{a}_-\hat{a}_+\psi_n = (n+1)\psi_n$

So,

$$\langle \hat{a}_+ \psi_n | \hat{a}_+ \psi_n \rangle = |c_n|^2 \langle \psi_{n+1} | \psi_{n+1} \rangle$$

$$= (n+1) \langle \psi_n | \psi_n \rangle$$

$$\Rightarrow \quad c_n = \sqrt{n+1}$$

and

$$\langle \hat{a}_- \psi_n | \hat{a}_- \psi_n \rangle = |d_n|^2 \langle \psi_{n-1} | \psi_{n-1} \rangle$$

$$= n \langle \psi_n | \psi_n \rangle \quad \Rightarrow \quad d_n = \sqrt{n}$$

So,

$$\boxed{\begin{aligned} \hat{a}_+ \psi_n &= \sqrt{n+1}\, \psi_{n+1} \\ \hat{a}_- \psi_n &= \sqrt{n}\, \psi_{n-1} \end{aligned}}$$

and

$$\boxed{\psi_n = \frac{1}{\sqrt{n!}} (\hat{a}_+)^n \psi_0}$$

These stationary states are orthogonal

$$\int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = \delta_{mn}$$

Pf: $\int_{-\infty}^{\infty} \psi_m^* (\hat{a}_+ \hat{a}_-) \psi_n \, dx = n \int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx$

$$= \int_{-\infty}^{\infty} (\hat{a}_- \psi_m)^* (\hat{a}_- \psi_n) \, dx = \int_{-\infty}^{\infty} (\hat{a}_+ \hat{a}_- \psi_m)^* \psi_n \, dx$$

$$= m \int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx \Rightarrow \int_{-\infty}^{\infty} \psi_m^* \psi_n \, dx = 0$$

unless $m = n$

In class probs: G 2.2 p30, G 2.1 p29, G 2.42 p86.

<u>Table</u> $\quad \hat{H} = \hat{p}^2/2m + \hat{V}$, $\quad V = \frac{1}{2} m \omega^2 x^2$

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m \omega}} (\mp i \hat{p} + m \omega \hat{x})$$

$$[\hat{a}_-, \hat{a}_+] = 1 \quad \text{and} \quad [\hat{x}, \hat{p}] = i\hbar$$

$$\hat{H} = \hbar \omega \left(\hat{a}_\pm \hat{a}_\mp \pm \tfrac{1}{2} \right)$$

$$\psi_n(x) = A_n (\hat{a}_+)^n \psi_0(x), \quad E_n = (n + \tfrac{1}{2}) \hbar \omega$$

$$\psi_0(x) = \left(\frac{m\omega}{\pi \hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}, \quad E_0 = \frac{\hbar \omega}{2}.$$