Quantum Mechanics Feb 27^{th}, 2015 Pl/4

Lect. 15

I Last time

II Free particle

III Wave packets

I. <u>Oscillator Table</u>

$$\hat{H} = \frac{\hat{P}^2}{2m} + \hat{V}, \qquad \hat{V} = \frac{1}{2} m \omega^2 \hat{x}^2$$

$$\hat{a}_\pm = \frac{1}{\sqrt{2\hbar m\omega}} \left[\mp i \hat{p} + m\omega \hat{x} \right]$$

$$[\hat{x}, \hat{p}] = i\hbar \quad \text{and} \quad [\hat{a}_-, \hat{a}_+] = 1$$

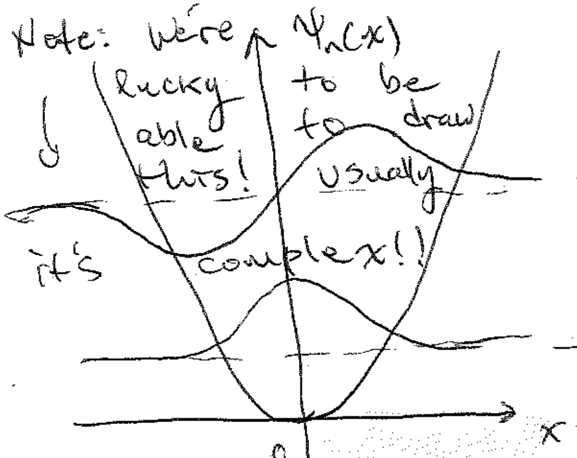
$$\hat{H} = \hbar\omega \left(\hat{a}_\pm \hat{a}_\mp \pm 1/2 \right)$$

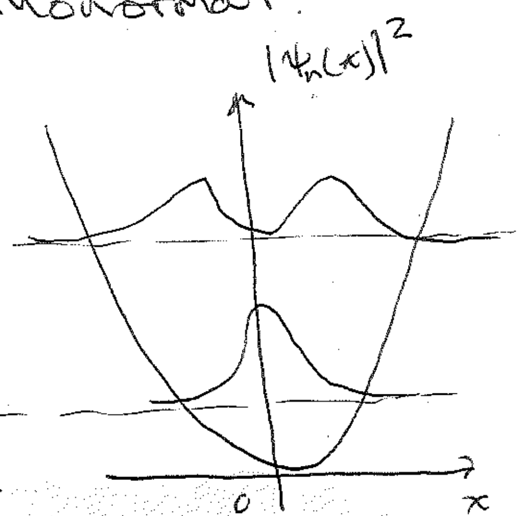
$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}, \quad E_0 = \frac{\hbar\omega}{2}$$

$$\psi_n(x) = \frac{1}{\sqrt{n!}} (\hat{a}_+)^n \psi_0(x), \quad E_n = (n+\tfrac{1}{2})\hbar\omega$$

The ψ_n are orthonormal.

Note: we're lucky to be able to draw this! Usually it's complex!!





II. What does it mean for a particle to be free? $V = 0$!

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi$$

or

$$\frac{d^2\psi}{dx^2} = -k^2 \psi, \quad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

Again the SHO diff. eqn. But, now ~~it's~~ better to write

$$\psi(x) = A e^{ikx} + B e^{-ikx}$$

Why the difference from the square well? No boundary conditions — so E's not quantized!

$$\Psi(x,t) = A e^{ik\left(x - \frac{\hbar k}{2m} t\right)} + B e^{-ik\left(x + \frac{\hbar k}{2m} t\right)}$$

↑ right traveling wave ↑ left mover

But any f s.t. $f = f(x \pm vt)$ is a wave moving in the $\mp x$ direction with speed v. (Note $x + vt = \text{const} \Rightarrow \frac{dx}{dt} = -v$)

The speed at which they move is

$$v_{quantum} = \frac{\hbar |k|}{2m} = \sqrt{\frac{E}{2m}}$$

Note that $\left(E = \frac{1}{2} m v^2\right)$

$$v_{classical} = \sqrt{\frac{2E}{m}} = 2 v_{quant}.$$

Apparently these stationary states are not, in themselves, modeling classical particles with a definite energy.

We can more simply write

$$\Psi_k(x,t) = A e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)}$$

with

$$k = \pm \frac{\sqrt{2mE}}{\hbar} \quad \begin{cases} k > 0 \Rightarrow \text{right mover} \\ k < 0 \Rightarrow \text{left mover} \end{cases}$$

Stationary states are waves with

$$\lambda = \frac{2\pi}{|k|} \quad \text{and} \quad p = \hbar k$$

Even worse Ψ_k is not normalizable:

$$|A|^2 \int_{-\infty}^{\infty} \Psi_k^* \Psi_k \, dx = |A|^2 \int_{-\infty}^{\infty} dx = \infty |A|^2.$$

These solutions are still immensely useful. As we have seen, they span the solutions of Sch.'s eqn.

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i\left(kx - \frac{\hbar k^2}{2m} t\right)} dk$$

III This <u>wave packet</u>, because it combines a range of k's, can be normalizable!

So, suppose I'm given $\Psi(x,0)$ that is normalizable, how do I find $\phi(k)$?

You found the answer when you proved Plancherel's theorem:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx}\, dk$$

$$\Updownarrow$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx$$

Then

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0)\, e^{-ikx}\, dx$$

and you can say it determines the shape of the wavepacket.

Why does combining wave numbers allow wave packets to become normalizable?

Let's explore through some examples:

<u>Ex 1</u>: Let's combine a finite number, say 3, plane waves. Let's choose k_0, $k_0 - \frac{\Delta k}{2}$, and $k_0 + \frac{\Delta k}{2}$ with amplitudes in ratio $1 : 1/2 : 1/2$ then

$$\Psi(x) = \frac{A}{\sqrt{2\pi}} \left[e^{ik_0 x} + \frac{1}{2} e^{i(k_0 - \frac{\Delta k}{2})x} + \frac{1}{2} e^{i(k_0 + \frac{\Delta k}{2})x} \right]$$

$$= \frac{A}{\sqrt{2\pi}} e^{ik_0 x} \left[1 + \cos\left(\frac{\Delta k}{2} x\right) \right]$$

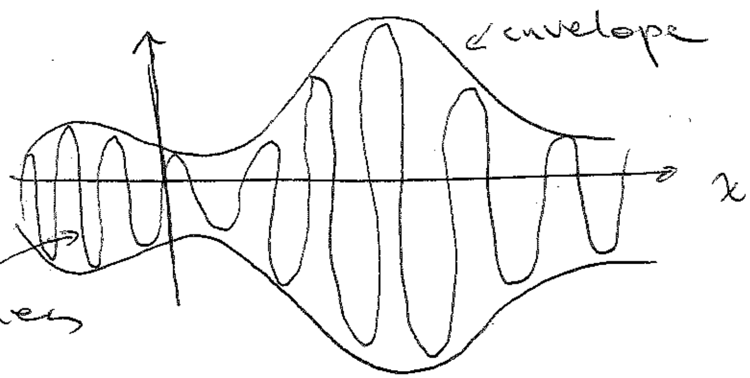
This is max at $x = 0$ but decreases as x increases due to deconstructive interference. The interference is completely deconstructive when the phase shift $e^{\pm i \frac{\Delta k}{2} x}$ is -1, i.e. when

$$\pm \frac{\Delta x}{2}\left(\pm \frac{\Delta k}{2}\right) = \pi$$

$$\Rightarrow \quad \Delta x \cdot \Delta k = 4\pi$$

This illustrates a remarkable (and completely general) trade off.

The structure of a wave packet is



The ripples travel at the <u>phase velocity</u>

$$v_{phase} = \frac{\omega}{k}$$

The more localized the wave packet (small Δx), the greater its width in wave numbers (large Δk, or Δp).

Tighter bounds relating these ranges will be our focus <u>next week.</u>

Didn't get past here until the next lecture.

But, what about the ~~group~~ envelope? It moves at the <u>group velocity</u>. Start with

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx-\omega t)}\, dk$$

[for us $\omega = \omega(k) = \frac{\hbar k^2}{2m}$, but the argument will work for all dispersion relations $\omega = \omega(k)$.]