Quantum Mechanics Lect. 16

Mar 2nd, 2015

0. Apologize, Curse of Knowledge, Good of the whole group.

I Last time

II Structure of wave packets

I • Studied the free particle Sch. eqn:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

• Found solutions of $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$

$$\Psi_k(x,t) = A e^{i(kx - \frac{\hbar k^2}{2m}t)}$$

with

$$k = \pm\frac{\sqrt{2mE}}{\hbar} \quad \begin{cases} k>0 \Rightarrow \text{right mover} \\ k<0 \Rightarrow \text{left mover} \end{cases}$$

However, these are not normalizable.

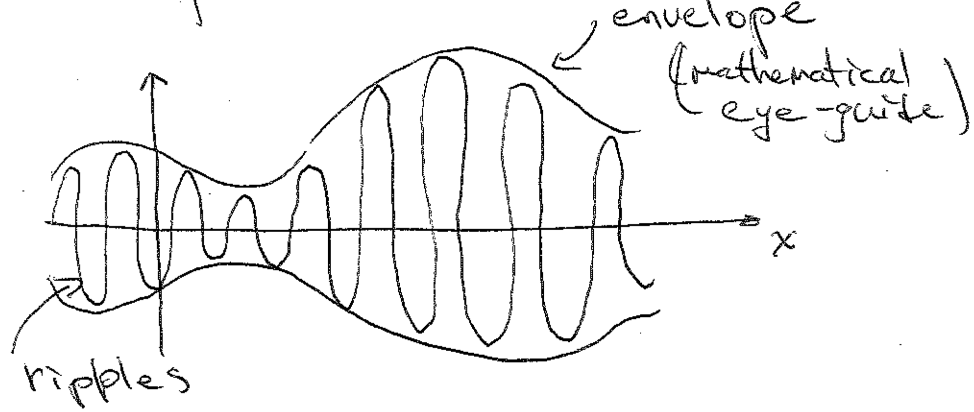
• So, build wave packets

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \phi(k)\, e^{i(kx - \underbrace{\frac{\hbar k^2}{2m}}_{\omega}t)}\, dk$$

$\phi(k)$ tells you how much of mode k is present.

II Wave packet structure and Speed

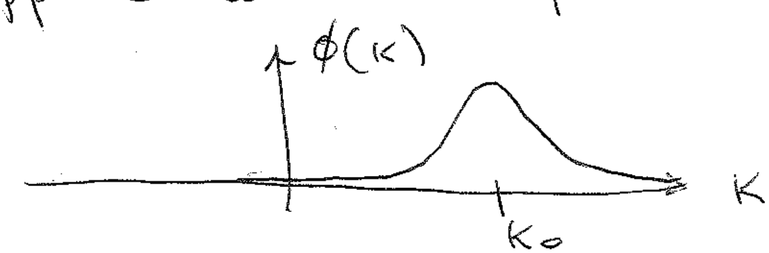
We left off last time with a simplified example that illustrated why mixing k's can lead to localization — the component waves have different wavelengths and can interfere deconstructively, effectively containing the packet. So, the structure of a wave packet is



[For us $\omega = \omega(k) = \frac{\hbar k^2}{2m}$, but the argument will work for all dispersion relations

$$\omega = \omega(k).]$$

Suppose we are peaked near k_0



The ripples travel at the <u>phase velocity</u>

$$v_{phase} = \frac{\omega}{k}$$

But, what about the envelope?

It moves at the <u>group velocity</u>. Start with

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k)\, e^{i(kx-\omega t)}\, dk$$

and expand

$$\omega(k) \simeq \omega_0 + \omega_0'(k-k_0) + \cdots$$

where $\omega_0 \equiv \omega(k_0)$ and $\omega_0' = \left.\frac{d\omega}{dk}\right|_{k_0}$

Let $s = k - k_0$, centering our integral at k_0

$$\Psi(x,t) \cong \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k_0+s) \times e^{i[(k_0+s)x - (\omega_0+\omega_0' s)t]}\, ds$$

Then

$$\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k_0+s)\, e^{i(k_0+s)x}\, ds$$

and later

$$\Psi(x,t) \cong \frac{1}{\sqrt{2\pi}} e^{i(-\omega_0 t + k_0\omega_0' t)} \times \int_{-\infty}^{\infty} \phi(k_0+s)\, e^{i(k_0+s)(x-\omega_0' t)}\, ds$$

the last is true since

$$\stackrel{?}{=} \frac{1}{\sqrt{2\pi}} e^{i(-\omega_0 t + k_0\omega_0' t)} \times \int_{-\infty}^{\infty} \phi(k_0+s)\, e^{i[(k_0+s)x - (k_0+s)\omega_0' t]}\, ds$$

But then,

$$\Psi(x,t) \cong \underbrace{e^{-i(\omega_0 - k_0\omega_0')t}}_{\text{just a phase}}\, \Psi(x-\omega_0' t, 0)$$

Aside from the overall phase this is a traveling wave with speed

$$v_{group} = \omega_0' = \left.\frac{d\omega}{dk}\right|_{k_0}$$

For our case

$$\omega = \frac{\hbar k^2}{2m}$$

and

$$\frac{d\omega}{dk} = \frac{\hbar k}{m}$$

and is indeed twice v_{phase}:

$$v_{group} = v_{classical} = 2\, v_{phase}.$$

[Recall, last time we found

$$v_{phase} = \frac{\omega}{k} = \frac{\hbar k}{2m}.$$]

This is an exciting mathematical indication that a quantum wave packet can be the underlying structure ~~of~~ for the classical particles so familiar to everyday life.

To confirm this hypothesis we turn again to experiment — there is compelling evidence to support it from the interference properties of Bucky balls (C_{60}) to measurements of the ~~...~~ internal wave packet structure of neutrons

<u>Extended example: The Gaussian Wave Packet</u>: A free particle has the initial wave function

$$\Psi(x,0) = A e^{-ax^2}$$

with A and a real constants, $a>0$.

(a) Normalize $\Psi(x,0)$

$$|A|^2 \int_{-\infty}^{\infty} e^{-ax^2} dx = |A|^2 \frac{\sqrt{\pi}}{\sqrt{a}} = 1$$

$$\Rightarrow \quad A = \left(\frac{a}{\pi}\right)^{1/4}.$$

(b) Find $\Psi(x,t)$, well

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A e^{-ax^2} e^{-ikx} dx$$

This integral is treated by turning it into a Gaussian:

$$ax^2 + bx = a\left(x^2 + \frac{b}{a}x\right)$$