Quantum Mechanics

Mar 4th, 2015

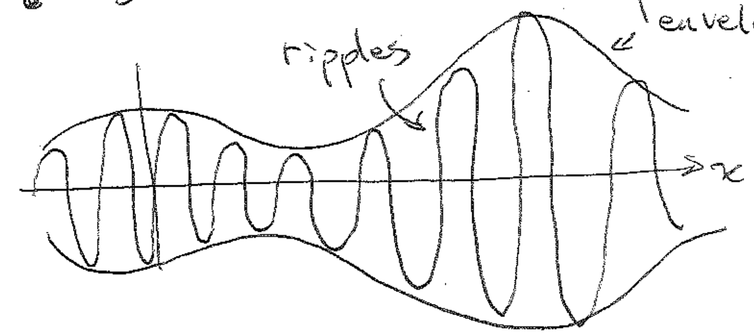
Lect. 17

I Last time
II What we've done up to ~~now~~
III Max's guest lecture:

Dirac notation & change of basis

~~III~~ IV Detailed discussion of the Gaussian wave packet.

- Structure of wave packets



- Phase velocity

$$v_{phase} = \frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = \lambda f.$$

- Group velocity

$$v_{group} = \left.\frac{d\omega}{dk}\right|_{k_0}$$

central wave #

Requires knowledge of the dispersion relation to evaluate

$$\omega = \omega(k)$$

dispersion relation

II We asked: why the 2-norm?
⇝ allows non-trivial transformations and interference phenomenon

Motivated the Schrödinger eqn as the promotion of Hamilton's ray theory of particles to a wave theory.

Developed some quantum formalism

$|\Psi\rangle = \alpha |\text{upper path}\rangle + \beta |\text{lower path}\rangle$

$|\alpha|^2$ = probability upper det. clicks

$|\beta|^2$ = " lower " "

$\Psi(x,t)$ = wave function

$\int_a^b |\Psi(x,t)|^2 dx$ = prob. finding particle between a and b.

For discrete spectra: $\hat{Q}$ is hermitian

q is real

Ψ_q provide an orthonormal set

The Ψ_q are complete (i.e. span $\mathcal{H}$).

The same statements hold morally in the continuum with the appropriate understandings

Hermitian Operators

$\boxed{\text{Observables are rep'd by hermitian operators}}$

Determinate states are eigenfunctions of hermitian operators; e.g.,

$$\hat{Q}\Psi_q = q\Psi_q$$

Got a bit abstract — so we turned to some examples:

Infinite Sq. well: solve Sch. eqn. directly.

Harmonic Osc.: solve using ladder operators

Free particle: solve directly, but interesting solutions are superpositions.

$\Longrightarrow$ Return to general theory.

III Max's guest lecture

IV Gaussian wave packets

(b)
$$\int_{-\infty}^{\infty} e^{-(ax^2+bx)}\,dx = \int_{-\infty}^{\infty} e^{-y^2+\left(\frac{b^2}{4a}\right)} \frac{1}{\sqrt{a}}\,dy$$

$$= \frac{1}{\sqrt{a}} e^{b^2/4a} \int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\frac{\pi}{a}}\, e^{b^2/4a}$$

Then

$$\phi(k) = \frac{1}{\sqrt{2\pi}} A \int_{-\infty}^{\infty} e^{-ax^2} e^{-ikx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2a}{\pi}\right)^{1/4} \sqrt{\frac{\pi}{a}}\; e^{-k^2/4a}$$

$$= \frac{1}{(2\pi a)^{1/4}}\, e^{-k^2/4a}$$

So,

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \frac{1}{(2\pi a)^{1/4}} \int_{-\infty}^{\infty} e^{-k^2/4a}\, e^{i\left(kx - \frac{\hbar k^2 t}{2m}\right)} dk$$

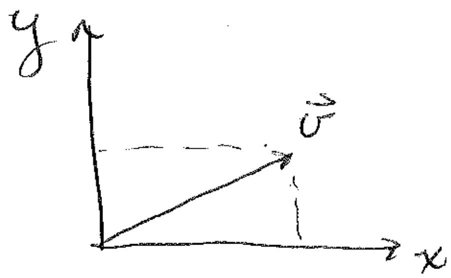
$$= \left(\frac{2a}{\pi}\right) \frac{e^{-\frac{ax^2}{(1+2i\hbar a t/m)}}}{\sqrt{1+2i\hbar a t/m}}$$

(c) Let $\theta = 2\hbar a t/m$ then

$$|\Psi|^2 = \sqrt{\frac{2a}{\pi}}\; \frac{e^{-a\frac{x^2}{(1+i\theta)}}\; e^{-\frac{ax^2}{(1-i\theta)}}}{\sqrt{(1+i\theta)(1-i\theta)}}$$

Max's Guest Lecture

Consider a vector $\vec{v}$,



$$\vec{v} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}$$

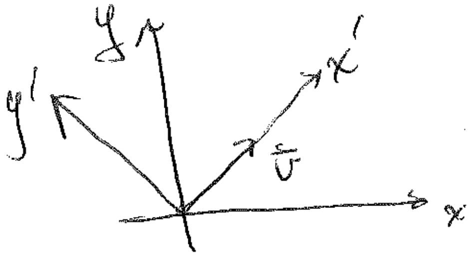
$$= \frac{1}{\sqrt{2}}|e_x\rangle + \frac{1}{\sqrt{2}}|e_y\rangle$$

In a more abstract way we can also write,

$$\vec{V} = \sum_i \langle e_i | v\rangle |e_i\rangle \quad \text{e.g. } i = x, y.$$

here we write $\vec{v}$ abstractly as $|v\rangle$.

If we choose a new basis we have



Call $\vec{v}'$ the vector $\vec{v}$ written in the new coordinates

$$\vec{V}' = \sum_{j=x'}^{y'} \sum_{i=x}^{y} \langle e_j | \langle e_i | v \rangle e_i \rangle | e_j \rangle$$

So

$$V_x' = \langle e_{x'} | \langle e_x | v \rangle e_x \rangle | e_{x'} \rangle + \langle e_{x'} | \langle e_y | v \rangle | e_y \rangle | e_{x'} \rangle$$

$$= | e_{x'} \rangle .$$

This compares nicely with what we know about Fourier Series

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \underbrace{e^{in\pi x/a}}_{\text{Basis } |e_n\rangle}$$

We can determine c_n using Fourier's trick

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, e^{-in\pi x/a} \, dx$$

$$= \langle e_n | f \rangle$$

A set of functions is said to be completie if

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

with

$$c_n = \langle f_n | f \rangle.$$

A quantum state is a vector in the Hilbert space $|V(t)\rangle$. Because we know that the eigenfunctions of hermitian operators span the Hilbert space, we can write $|V(t)\rangle$ in the basis of this set of functions.

So,

position eigenstates

$$\Psi(x,t) = \langle f_x | V(t) \rangle$$

But, we could also write

$$\Phi(p,t) = \langle f_p | V(t) \rangle.$$ "momentum space wavefunction."

Here

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}, \quad \hat{p} f_p = p f_p.$$