Quantum Mechanics

Lect. 18

I Last time

II General statistical interpretation

III The Uncertainty Principle

— physical vectors are independent of the basis that we describe them in.

— We can transition from one basis, say $\{|e_n\rangle\}_{n=1}^{N}$, to another, say $\{|e_n'\rangle\}_{n=1}^{N}$

~~using the overlaps~~

$$|V\rangle = \langle e_1|V\rangle\,|e_1\rangle + \underbrace{\langle e_2|V\rangle}_{\text{number expressing component}}\,\underbrace{|e_2\rangle}_{\text{vector in } e_2\text{-direction}} + \dots + \langle e_n|V\rangle\,|e_n\rangle$$

I • Reviewed whole class to present and noted that we are returning to the general theory

• Max introduced us to several ideas:

$$|V\rangle = \langle e_1'|V\rangle\,|e_1'\rangle + \dots + \langle e_n'|V\rangle\,|e_n'\rangle$$

↑ same vector $|V\rangle$.

• We can also do this with countably infinite bases

$$f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

with

$$c_n = \langle f_n | f \rangle.$$

The fact that this change of basis doesn't affect the vector means that we could just as well describe

$$\Psi(x,t) = \langle f_x | V(t) \rangle$$

in the momentum basis

$$\Phi(p,t) = \langle f_p | V(t) \rangle$$

Represents the same physical system

Now,

$$\int_{p_1}^{p_2} |\Phi(p,t)|^2 \, dp = \left\{ \begin{array}{l} \text{prob. of finding particle} \\ \text{with momentum} \end{array} \right.$$

between p_1 and p_2,

II. Suppose $\hat{Q}$ hermitian with a discrete spectrum. If we measure $\hat{Q}$ in state $\Psi(x,t)$ we will get one of the eigenvalues q of $\hat{Q}$.

What is the probability of q_n?

Let $f_n(x)$ be the n^{th} e-function of $\hat{Q}$ then

$$\text{Prob.} = |c_n|^2 \quad \text{where } c_n = \langle f_n | \Psi \rangle.$$

For continuous spectrum

$$\underset{\text{between } z \text{ and } z+dz}{\text{Prob}} = |c(z)|^2 dz \quad \text{where } c(z) = \langle f_z | \Psi \rangle.$$

The definitions of c_n and $c(z)$ are ~~just~~ really just applications of Fourier's trick:

$$\Psi(x,t) = \sum_n c_n f_n(x)$$

← completeness

$$\Rightarrow c_n = \langle f_n | \Psi \rangle.$$

We must have

$$\sum_n |c_n|^2 = 1$$

and

$$\langle Q \rangle = \sum_n q_n |c_n|^2$$

Want to prove one?

$$1 = \langle \Psi | \Psi \rangle = \langle \sum_{n'} c_{n'} f_{n'} | \sum_n c_n f_n \rangle$$

$$= \sum_{n,n'} c_{n'}^* c_n \langle f_{n'} | f_n \rangle$$

$$= \sum_{n,n'} c_{n'}^* c_n \delta_{n'n} = \sum_n c_n^* c_n = \sum_n |c_n|^2 \checkmark$$

Let's apply this to learn what $\Phi(p,t)$ is in terms of $\Psi(x,t)$:

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p)\, f_p(x)\, dp$$

We have

$$c(p) = \langle f_p | \Psi \rangle$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} f_p^*(x)\, \Psi(x,t)\, dx$$

$$= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar}\, \Psi(x,t)\, dx \equiv \Phi(p,t)$$

and so

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{ipx/\hbar}\, \Phi(p,t)\, dp$$

The position and momentum space wave functions are just Fourier transforms of one another!

III You've checked it lots: $\sigma_x \sigma_p \geq \frac{\hbar}{2}$

Let's prove it!

Consider observable A

$$\sigma_A^2 \equiv \langle (\hat{A} - \langle A \rangle)\Psi | (\hat{A} - \langle A \rangle)\Psi \rangle$$

$$\equiv \langle f | f \rangle$$

with $f = (\hat{A} - \langle A \rangle)\Psi$

Similarly,

$$\sigma_B^2 = \langle g|g\rangle \quad \text{w/} \quad g = (\hat{B} - \langle B\rangle)\Psi$$

Now, we apply the Schwarz ineq.

$$\left|\int_a^b f^* g \, dx\right| \le \sqrt{\int_a^b |f|^2 dx \int_a^b |g|^2 dx}$$

to find

$$\sigma_A^2 \sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \ge |\langle f|g\rangle|^2$$

Now, compute

$$\langle f|g\rangle = \langle(\hat{A} - \langle A\rangle)\Psi | (\hat{B} - \langle B\rangle)\Psi\rangle$$

$$= \langle \Psi | (\hat{A} - \langle A\rangle)(\hat{B} - \langle B\rangle)\Psi\rangle$$

$$= \langle \Psi | [\hat{A}\hat{B} - \langle A\rangle\hat{B} - \langle B\rangle\hat{A} + \langle A\rangle\langle B\rangle]\Psi\rangle$$

$$= \langle \Psi | \hat{A}\hat{B}\Psi\rangle - \langle A\rangle\langle B\rangle - \langle B\rangle\langle A\rangle + \langle A\rangle\langle B\rangle$$

$$= \langle \hat{A}\hat{B}\rangle - \langle A\rangle\langle B\rangle.$$

$$\langle g|f\rangle = \langle \hat{B}\hat{A}\rangle - \langle A\rangle\langle B\rangle \quad \text{[check it!]}$$

Also, $z = a + ib$

$$|z|^2 = a^2 + b^2$$

$$= (\mathrm{Re}(z))^2 + (\mathrm{Im}(z))^2$$

$$\ge (\mathrm{Im}(z))^2 = \left[\frac{1}{2i}(z - z^*)\right]^2$$

So, if we let $z = \langle f|g\rangle$,

$$\sigma_A^2 \sigma_B^2 \ge \left(\frac{1}{2i}[\langle f|g\rangle - \langle g|f\rangle]\right)^2$$

so

$$\langle f|g\rangle - \langle g|f\rangle = \langle \hat{A}\hat{B}\rangle - \langle \hat{B}\hat{A}\rangle$$

$$= \langle \Psi | (\hat{A}\hat{B} - \hat{B}\hat{A}) | \Psi\rangle$$

$$= \langle [\hat{A}, \hat{B}]\rangle$$

and

$$\sigma_A^2 \sigma_B^2 \ge \left(\frac{1}{2i}\langle [\hat{A}, \hat{B}]\rangle\right)^2$$

Let's try it:

$$[\hat{x}, \hat{p}] = i\hbar$$

So

$$\sigma_x^2 \sigma_p^2 \geq \left(\frac{1}{2i} i\hbar\right)^2 = \frac{\hbar^2}{4}$$

and we can conclude

$$\boxed{\sigma_x \sigma_p \geq \frac{\hbar}{2}}$$