Quantum Mechanics

Lect. 19

Mar 9th, 2015 P1/5

I Last time

II Interpreting "uncertainty" and the energy-time uncertainty principle.

• Derived a precise statement of the uncertainty principle

$$\sigma_A^2 \sigma_B^2 \geq \left(\frac{1}{2i} \langle [\hat{A}, \hat{B}] \rangle \right)^2$$

• Applied this to find

$$\sigma_x \sigma_p \geq \frac{\hbar}{2}$$

• Incompatible observables: any pair of observables that don't commute is said to be incompatible. Such observables cannot be simultaneously determinate.

I • General statistical interpretation

If we measure $\hat{Q}$ on $\Psi(x,t)$ we will get an eigenvalue q with probability

Prob. $= |c_n|^2$ with $c_n = \langle f_n | \Psi \rangle$

II The uncertainty principles require care in their interpretation

We often write

$$\underbrace{\Delta x}\, \Delta p \geq \frac{\hbar}{2}$$

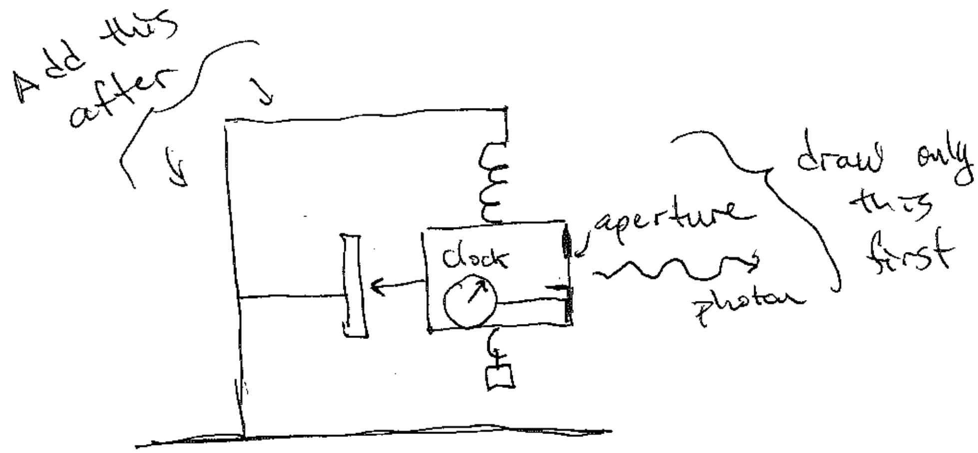
"uncertainty" in x $\rightsquigarrow$ more precisely σ_x

where σ_x is the standard deviation of the results of repeated measurements on identically prepared systems

You may not be surprised to learn that there is a similar relation

$$\Delta t \, \Delta E \geq \frac{\hbar}{2}$$

Einstein proposed an experiment to refute such a relationship



Open shutter for time Δt

This seems plausible on the grounds that x and t and p and E play similar parallel roles in special relativity. But, there is no t operator in ~~special~~ our framework (non-rel. Q.M.).

So, what can this mean?

Use $E = mc^2$ to determine E to precision ΔE. It seems that you can make the product

$$\Delta E \, \Delta t$$

arbitrarily small.

Read Rosenfeld quote about Bohr's reaction (e.g. off Wikipedia)

Bohr's equivalence principle argument $\Rightarrow$ $\Delta E \Delta t \geq h$

So there are technical and conceptual challenges. Here's a formulation I like a lot: consider $\hat{Q}(x,p,t)$

$$\frac{d\langle Q\rangle}{dt} = \frac{d}{dt}\langle \Psi | \hat{Q}\Psi\rangle$$

$$= \left\langle \frac{\partial \Psi}{\partial t} \middle| \hat{Q}\Psi\right\rangle + \left\langle \Psi \middle| \frac{\partial \hat{Q}}{\partial t}\Psi\right\rangle + \left\langle \Psi \middle| \hat{Q}\frac{\partial \Psi}{\partial t}\right\rangle$$

But, $i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$ and so,

$$\frac{d\langle Q\rangle}{dt} = -\frac{1}{i\hbar}\langle \hat{H}\Psi | \hat{Q}\Psi\rangle + \frac{1}{i\hbar}\langle \Psi | \hat{Q}\hat{H}\Psi\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle$$

Then, since $\hat{H}^\dagger = \hat{H}$ we get

$$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H},\hat{Q}]\rangle + \left\langle \frac{\partial \hat{Q}}{\partial t}\right\rangle.$$

IF, as often happens $\frac{\partial \hat{Q}}{\partial t} = 0$,

$$\frac{d\langle Q\rangle}{dt} = \frac{i}{\hbar}\langle [\hat{H},\hat{Q}]\rangle.$$

So, when $[\hat{H},\hat{Q}]=0$ the expectation value $\langle \hat{Q}\rangle$ is conserved!

Then let's pick $\hat{A} = \hat{H}$ and $\hat{B} = \hat{Q}$, we find

$$\sigma_H^2 \sigma_Q^2 \geq \left(\frac{1}{2i}\langle [\hat{H},\hat{Q}]\rangle\right)^2$$

$$= \left(\frac{1}{2i}\frac{\hbar}{i}\frac{d\langle Q\rangle}{dt}\right)^2$$

$$= \left(\frac{\hbar}{2}\right)^2\left(\frac{d\langle Q\rangle}{dt}\right)^2$$

or

$$\sigma_H \sigma_Q \geq \frac{\hbar}{2}\left|\frac{d\langle Q\rangle}{dt}\right|$$

If we <u>define</u> $\Delta E = \sigma_H$ and

$$\Delta t = \frac{\sigma_Q}{\left|\frac{d\langle Q\rangle}{dt}\right|}$$

then

$$\Delta E\,\Delta t \geq \frac{\hbar}{2}. \checkmark$$

It's essential here that we

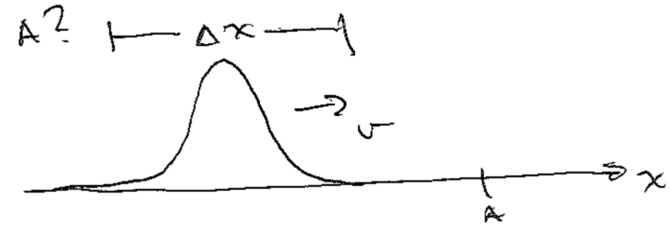
understand that Δt measure the time that it takes $\langle Q \rangle$ to change by one standard deviation:

$$\sigma_Q = \left| \frac{d\langle Q \rangle}{dt} \right| \Delta t.$$

This is the Mandelstam–Tamm formulation.

You could think of $\hat{Q}$ as the practical implementation of our clock, Δt measures the shortest time scale on which we can notice sizable changes of $\hat{Q}$ in the state Ψ.

Ex: How long does it take a wave packet to pass A?

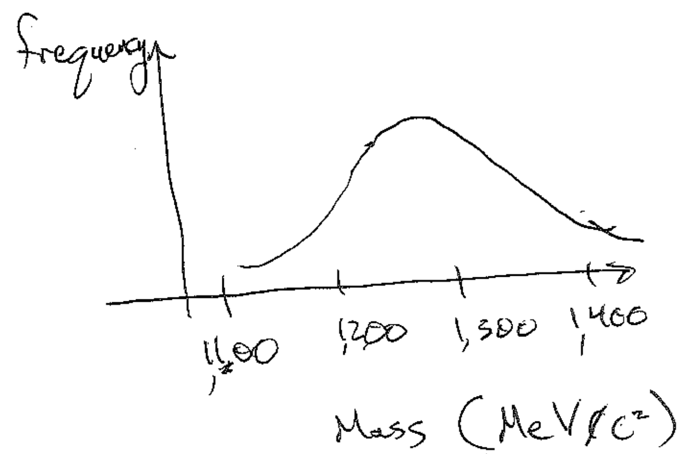


Rough estimate: $\Delta t = \frac{\Delta x}{v} = \frac{m \Delta x}{p}$

and $E = \frac{p^2}{2m}$ so $\Delta E = \frac{p \Delta p}{m}$, so

$$\Delta E \Delta t = \frac{p \Delta p}{m} \frac{m \Delta x}{p} = \Delta x \Delta p \geq \hbar/2$$

Ex 2: A nice example is the decay of the Δ particle. The lifetime of the Δ is $\sim 10^{-23}$ secs. If you measure the mass it is centered



at $1232\ MeV/c^2$ with a width $\sim 120\ MeV/c^2$. Why this variation in mc^2? Error?

No!

$$\Delta E \Delta t = \left(\frac{120}{2}\,\text{MeV}\right)\left(10^{-23}\,\text{sec}\right)$$

$$= 6 \times 10^{-22}\,\text{MeV}\cdot\text{sec}$$

while

$$\frac{\hbar}{2} = 3 \times 10^{-22}\,\text{MeV sec} \,!$$

The spread is as small as $\Delta E \Delta t$ allows!

A particle with this short a lifetime doesn't have a well-defined mass!!