Quantum Mechanics

Mar 23, 2015 P1/3

Lect. 22

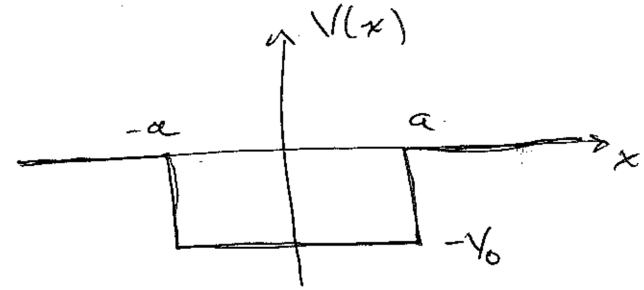
I Feedback Summary

II Exam summary & discussion of one problem

III Finite square well

III One more 1D example

$$V(x) = \begin{cases} -V_0 & -a \le x \le a \\ 0 & |x| > a \end{cases}$$



Bound States: For $x < -a$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad \text{or} \quad \frac{d^2\psi}{dx^2} = k^2\psi$$

with $$k = \frac{\sqrt{-2mE}}{\hbar} \quad (\text{recall } E<0)$$

$$\psi(x) = A e^{-kx} + B e^{kx}$$

But as $x \to -\infty$ e^{-kx} blows up, so $A=0$ and

$$\psi(x) = B e^{kx} \quad x < -a.$$

For $-a < x < a$ $V(x) = -V_0$ and

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi$$

or $$\frac{d^2\psi}{dx^2} = -l^2\psi$$

with $$l = \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

recall $E > V_{min} = V_0$ (proved on HW)

so $l \in \mathbb{R}^+$

$$\psi(x) = C\sin(lx) + D\cos(lx) \quad -a<x<a$$

For $x \geq a$, $V=0$ again and

$$\psi(x) = F e^{-\kappa x} + G e^{\kappa x}$$

Now, the growing $\exp(\kappa x)$ ~~[illegible]~~ must be killed so

$$\psi(x) = F e^{-\kappa x} \qquad x > a$$

Next we impose boundary conditions

ψ and $\frac{d\psi}{dx}$ are continuous

At a

$$F e^{-\kappa a} = D \cos(la)$$

and

$$-\kappa F e^{-\kappa a} = -l D \sin(la)$$

Then dividing implies

$$\kappa = l \tan(la)$$

Let's simplify this a bit $z = la$ and $z_0 \equiv \frac{a}{\hbar}\sqrt{2mV_0}$. Now $\kappa^2 + l^2 = \frac{2mV_0}{\hbar^2}$

But since the potential is even, we can look for even or odd solutions — a general solution is built out of these.

Let's do the even ones:

$$\psi(x) = \begin{cases} F e^{-\kappa x} & x > a \\ D\cos(lx) & 0 < x < a \\ \psi(-x) & \text{for } x < 0 \end{cases}$$

and then

$$\kappa = \sqrt{\frac{2mV_0}{\hbar^2} - l^2}$$

or

$$\kappa a = \sqrt{z_0^2 - z^2}$$

Putting this into the $\tan(la)$ relation gives

$$\tan(z) = \frac{\kappa a}{la} = \sqrt{(z_0/z)^2 - 1}$$

This is a transcendental equation — it tells you how many bound E's there are given the width a and depth V_0 of the well.

Some limiting cases: <u>Wide, deep well:</u>

z_0 large $\Rightarrow \tan(z)$ large which is when $z \approx \frac{n\pi}{2} \equiv z_n$ then

$$z_n = l_n a = \frac{a\sqrt{2m(E_n+V_0)}}{\hbar} = \frac{n\pi}{2}$$

or

$$\boxed{E_n + V_0 = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}}$$

These are the square well solutions above V_0. But only works for finitely many n.

<u>Shallow, narrow well:</u>

As z_0 decreases there are fewer and fewer bound states, but always one remains!