Quantum Mechanics

Lect. 25

Mar 30^{th}, 2015 P1/4

I What we've learned about angular momentum

II Find the angular momentum eigenfunctions

I. $L_z = xP_y - yP_x$ etc.

$$L_\pm = L_x \pm i L_y$$

$$[L_x, L_y] = i\hbar L_z \text{ and } \circlearrowright$$

$$[L^2, L_z] = 0 \text{ and } x \text{ and } y$$

$$[L_z, L_\pm] = \pm \hbar L_\pm$$

We found

$$L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z$$

and

$$L^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m$$

$$L_z f_\ell^m = \hbar m f_\ell^m$$

Where

$$\ell = 0, 1/2, 1, 3/2, \ldots$$

$$m = -\ell, -\ell+1, \ldots, \ell-1, \ell.$$

II Classically $\vec{L} = \vec{r} \times \vec{p}$. As an operator

$$\hat{\vec{L}} = \frac{\hbar}{i} \vec{r} \times \vec{\nabla}.$$

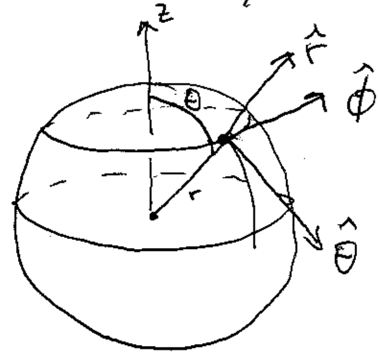
In spherical coords (natural for ang. momentum and rotations)

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Now, $\vec{r} = r\hat{r}$, so

$$\vec{L} = \frac{\hbar}{i}\left[r(\hat{r}\times\hat{r})\frac{\partial}{\partial r} + (\hat{r}\times\hat{\theta})\frac{\partial}{\partial\theta} + \frac{(\hat{r}\times\hat{\phi})}{\sin\theta}\frac{\partial}{\partial\phi}\right]$$

$$= \frac{\hbar}{i}\left[\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]$$



Then

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

while

$$L_x = \frac{\hbar}{i}\left(-s\phi\frac{\partial}{\partial\theta} - \cot\theta\, c\phi\frac{\partial}{\partial\phi}\right)$$

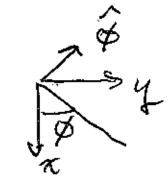
$$L_y = \frac{\hbar}{i}\left(c\phi\frac{\partial}{\partial\theta} - \cot\theta\, s\phi\frac{\partial}{\partial\phi}\right)$$

The raising and lowering operators are slightly simpler

Cartesian components will also be useful

$$\hat{\theta} = c\theta c\phi\,\hat{x} + c\theta s\phi\,\hat{y} - s\theta\,\hat{z}$$

$$\hat{\phi} = -s\phi\,\hat{x} + c\phi\,\hat{y}$$



So,

$$\vec{L} = \frac{\hbar}{i}\left[(-s\phi\,\hat{x} + c\phi\,\hat{y})\frac{\partial}{\partial\theta} - (c\theta c\phi\,\hat{x} + c\theta s\phi\,\hat{y} - s\theta\,\hat{z})\frac{1}{s\theta}\frac{\partial}{\partial\phi}\right]$$

$$L_\pm = L_x \pm iL_y$$

$$= \frac{\hbar}{i}\left[(-s\phi \pm ic\phi)\frac{\partial}{\partial\theta} - (c\phi \pm is\phi)\cot\theta\frac{\partial}{\partial\phi}\right]$$

$$= \pm\hbar\, e^{\pm i\phi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\frac{\partial}{\partial\phi}\right)$$

From which

$$L_+L_- = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \cot^2\theta\frac{\partial^2}{\partial\phi^2} + i\frac{\partial}{\partial\phi}\right)$$

and

$$L^2 = -\hbar^2\left[\frac{1}{s\theta}\frac{\partial}{\partial\theta}\left(s\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{s^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

Now,

$$L^2 f_\ell^m = \hbar^2 \ell(\ell+1) f_\ell^m$$

where $f_\ell^m = f_\ell^m(\theta,\phi)$. Let's suppose (separation of variables)

$$f_\ell^m(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)$$

to find

$$-\hbar^2\left[\frac{1}{s\theta}\frac{\partial}{\partial\theta}\left(s\theta\,\Phi\,\frac{d\Theta}{d\theta}\right) + \frac{1}{s^2\theta}\,\Theta\,\frac{d^2\Phi}{d\phi^2}\right] = \hbar^2 \ell(\ell+1)\Theta\Phi$$

Dividing by $\Theta\Phi$ and multiplying by $-s^2\theta$ gives

$$\frac{1}{\Theta}\left[s\theta\frac{d}{d\theta}\left(s\theta\frac{d\Theta}{d\theta}\right)\right] + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -\ell(\ell+1)s^2\theta$$

or

$$\left\{\frac{1}{\Theta}\left[s\theta\frac{d}{d\theta}\left(s\theta\frac{d\Theta}{d\theta}\right)\right] + \ell(\ell+1)s^2\theta\right\} + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0$$

But then $\{\cdots\}$ and $\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$ must each equal a constant

and

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2 \quad \text{(1)}$$

$\leftarrow$ call it

$$\{\cdots\} = m^2 \quad \text{(2)}$$

The first is easy to solve

$$\Phi(\phi) = e^{im\phi}$$

The "boundary" condition $\Phi(\phi+2\pi) = \Phi(\phi)$ yields

$$e^{im(\phi+2\pi)} = e^{im\phi} \Rightarrow e^{im2\pi} = 1 \Rightarrow m = 0, \pm1, \pm2, \ldots$$

Eq. (2) is more complicated

$$s\theta\frac{d}{d\theta}\left(s\theta\frac{d\Theta}{d\theta}\right) + \left[\ell(\ell+1)s^2\theta - m^2\right]\Theta = 0$$

and has solution

$$\Theta(\theta) = A\,P_\ell^m(\cos\theta)$$

where P_ℓ^m is the "associated Legendre function"

$$P_\ell^m(x) = (1-x^2)^{|m|/2}\left(\frac{d}{dx}\right)^{|m|} P_\ell(x),$$

and $P_\ell(x)$ is the ℓ^{th} Legendre Polynomial,
and is defined by

$$P_\ell(x) \equiv \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell$$

This is called the Rodrigues formula.