Quantum Mechanics

Apr 3rd, 2015

Lecture 27

I Last time

II Using the spin formalism

III A dialogue on spin measurement

IV Spins in magnetic fields

I. • Michael introduced Spin:

$$[S_x, S_y] = i\hbar S_z \quad \text{and} \ \circlearrowleft$$

$$S^2 |s, m\rangle = \hbar^2 s(s+1) |s, m\rangle$$

$$S_z |s, m\rangle = \hbar m |s, m\rangle$$

$$S_\pm = S_x \pm i S_y$$

• Different notations for the same thing:

$|\tfrac{1}{2}\ \tfrac{1}{2}\rangle$ $\uparrow$ (spin up) $\chi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle$ $\downarrow$ (spin down) $\chi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

General Spinor

$$a|\tfrac{1}{2}\ \tfrac{1}{2}\rangle + b|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle \text{ or } \chi = a\chi_+ + b\chi_- = \begin{pmatrix} a \\ b \end{pmatrix}$$

• Results

$$S^2 \chi_+ = \tfrac{3}{4}\hbar^2 \chi_+, \quad S^2 \chi_- = \tfrac{3}{4}\hbar^2 \chi_-$$

$$\Rightarrow \quad S^2 = \tfrac{3}{4}\hbar^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_z \chi_+ = \tfrac{\hbar}{2} \chi_+, \quad S_z \chi_- = -\tfrac{\hbar}{2} \chi_-$$

$$S_x = \tfrac{1}{2}(S_+ + S_-)$$

$$S_y = \tfrac{1}{2i}(S_+ - S_-)$$

II Derive the matrices for

$$S^2, S_x, S_y, S_z, S_+, S_-.$$

Find the eigenvalues and eigenvectors of the matrix (group of two)

$$\begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}$$

The eigenvalues are found using

$$\begin{vmatrix} -\lambda & \hbar/2 \\ \hbar/2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \Rightarrow \lambda = \pm \hbar/2$$

Then, after dropping overall phase,

$$\chi_+^{(x)} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \text{ and } \chi_-^{(x)} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

What's the point? If we measure S_x of a particle the possible outcomes are

$$\lambda_+ = +\hbar/2 \text{ or } \lambda_- = -\hbar/2$$

The states $\chi_+^{(x)}$ and $\chi_-^{(x)}$ have

$$S_x \chi_+^{(x)} = +\hbar/2\, \chi_+^{(x)} \text{ etc.}$$

The e-vectors are

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm \frac{\hbar}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \pm \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

So $\beta = \pm\alpha$. Next normalize, say $\beta = +\alpha$,

$$(\alpha^* \ \alpha^*)\begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = 2|\alpha|^2 = 1$$

$$\Rightarrow \alpha = \frac{1}{\sqrt{2}} e^{i\eta}$$

Note that $\chi_+^{(x)}$ and $\chi_-^{(x)}$ still provide a basis for $\mathcal{H}_{1/2}$.

Consider

$$\chi = a\chi_+ + b\chi_- = \begin{pmatrix} a \\ b \end{pmatrix}$$

We can also write this as

$$\chi = \left(\frac{a+b}{\sqrt{2}}\right)\chi_+^{(x)} + \left(\frac{a-b}{\sqrt{2}}\right)\chi_-^{(x)}$$

check it

$$\stackrel{?}{=} \left(\frac{a+b}{2}\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \left(\frac{a-b}{2}\right)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} a \\ b \end{pmatrix} \checkmark$$

So, what are the probabilities of measuring $+\hbar/2$ and $-\hbar/2$ in the x-direction when $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$?

Use the coeffs of $\chi_+^{(x)}$ and $\chi_-^{(x)}$!

$$\text{Prob.}(+\hbar/2) = \tfrac{1}{2}|a+b|^2$$

$$\text{Prob}(-\hbar/2) = \tfrac{1}{2}|a-b|^2$$

III ~~Choose a simple state,~~

~~Say χ_+.~~

~~What is the z-component of the particles spin angular momentum?~~

Your text has a very nice dialogue pp 176-177. Read this as many times as it takes for it to make perfect sense — there will be an oral quiz on it on Monday.

IV A spinning charged particle has a magnetic dipole moment $\vec{\mu}$:

$$\vec{\mu} = \gamma \vec{S}$$

here the constant γ is called the gyromagnetic ratio

When put in a magnetic field

Such a particle has energy

$$H = -\vec{\mu} \cdot \vec{B}$$

$$= -\gamma \vec{S} \cdot \vec{B}$$

A gyroscope in a gravitational field precesses — quite similarly a quantum spin in a magnetic field also precesses. ~~This is~~