Quantum Mechanics April 10th 2015 P1/4

Lecture 30

I Last time

II Q.M. in 3D

III Central potentials

IV The radial equation

I. Derived the Stern-Gerlach outcome quantum mechanically
→ we found that after traversing the inhomogeneous field the ~~Spin up~~ atoms had

$$P_z = + \frac{\alpha \gamma T \hbar}{2} \quad \text{for } \uparrow$$

and

$$P_z = - \frac{\alpha \gamma T \hbar}{2} \quad \text{for } \downarrow$$

So the beam splits.

- Practiced with the SPINS simulation ⇝ e.g. showed superposition.

II Return to the general Spatial Schrödinger equation today

$$i\hbar \frac{\partial \Psi}{\partial t} = H \Psi$$

where

$$H = \frac{1}{2m}\left(P_x^2 + P_y^2 + P_z^2\right) + V$$

We proceed with the standard rule

$$P_x \to \frac{\hbar}{i}\frac{\partial}{\partial x}, \quad P_y \to \frac{\hbar}{i}\frac{\partial}{\partial y}, \quad P_z \to \frac{\hbar}{i}\frac{\partial}{\partial z}$$

or, more succinctly,

$$\vec{p} \longrightarrow \frac{\hbar}{i}\vec{\nabla}$$

Then

$$i\hbar\frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + V\Psi$$

and $d^3\vec{r} = dx\,dy\,dz$.

Recall that if $V = V(\vec{r})$ then

$$\Psi_n(\vec{r},t) = \psi_n(\vec{r})\,e^{-iE_nt/\hbar}$$

is a complete set of stationary states and we can turn to study of

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

where,

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is the "Laplacian op.".

As in 1D we interpret

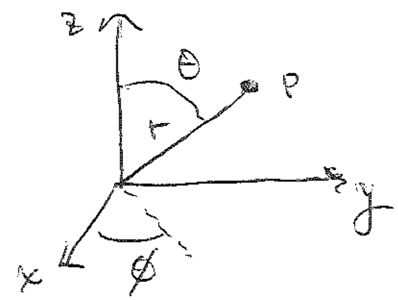
$$|\Psi(\vec{r},t)|^2 d^3\vec{r} = \left\{ \begin{array}{l} \text{Prob. of} \\ \text{finding the} \\ \text{particle in} \\ \text{volume } d^3\vec{r} \end{array} \right.$$

where, $\vec{r} = (x, y, z)$

III An interesting case to study is one where

$$V(\vec{r}) = \underset{\substack{\uparrow \\ \text{only depends on radial distance}}}{V(r)}$$

$r = |\vec{r}|$.



In these spherical coordinates

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 s\theta}\frac{\partial}{\partial\theta}\left(s\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2}\frac{1}{s^2\theta}\frac{\partial^2}{\partial\phi^2}$$

~~Let's~~ Notice that part of this is exactly

$$-\frac{1}{\hbar^2 r^2}L^2$$

So, let's look for solutions $\psi(r,\theta,\phi)$ s.t.

$$\psi(r,\theta,\phi) = R(r)Y(\theta,\phi).$$

Then

$$-\frac{\hbar^2}{2m}\nabla^2(RY) + V\overset{(RY)}{\cancel{\psi}} = E(RY)$$

$$\Rightarrow -\frac{\hbar^2}{2mr^2}Y\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\hbar^2}{\cancel{\hbar^2}2mr^2}RL^2\cancel{Y} + VRY = ERY$$

Dividing by RY and multiplying by $-2mr^2/\hbar^2$:

$$\left\{\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r)-E\right]\right\} + \frac{1}{Y}\left\{\frac{1}{s\theta}\frac{\partial}{\partial\theta}\left(s\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{s^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right\} = 0$$

so

$$\frac{1}{-\hbar^2 Y}L^2 Y = -l(l+1)$$

or $\boxed{L^2 Y = l(l+1)\hbar^2 Y}$

and

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2mr^2}{\hbar^2}\left[V(r)-E\right] = l(l+1)$$

We've studied the angular equation extensively.

So, let's study the radial equation.

Change variables to $u(r) = rR(r)$

then $R = u/r$ $\quad \frac{dR}{dr} = [r\frac{du}{dr} - u]/r^2$

and $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = r^2\frac{d^2u}{dr^2}$

So that

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu}$$

This looks just like the Schrödinger equation except with a new potential

$$V_{eff} = V + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}$$

The normalization condition is also only for positive

r now:

$$\int_0^\infty |u|^2 dr = 1.$$

To go further we have to pick a specific potential $V(r)$.