Quantum Mechanics April 17^{th}, 2015 P1/4

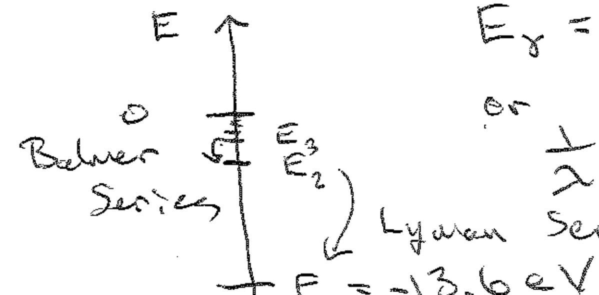
Lect. 33

I Last time

II Harmonic oscillator by power series

III Return to the hydrogen atom

I • Maya told us about atomic transitions:



$$E_\gamma = -13.6\,eV\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right)$$

or

$$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

$R = 1.097 \times 10^7\,m^{-1}$

• Started to review the series approach to solving differential eqn.'s:

- Find asymptotic behavior
- Strip off asymp. behavior
- Take ansatz

$$\psi = (\text{asymp. behavior}) \cdot (\text{power series})$$

• For the harmonic oscillar we had found

$$\psi(\xi) = h(\xi)\, e^{-\xi^2/2}$$

and the Sch. eqn:

$$\frac{d^2h}{d\xi^2} - 2\xi\frac{dh}{d\xi} + (K-1)h = 0$$

Now, we search for solutions of the form

$$h(\xi) = \sum_{j=0}^{\infty} a_j \xi^j = a_0 + a_1\xi + a_2\xi^2 + \cdots$$

$$\frac{dh}{d\xi} = a_1 + 2a_2\xi + \cdots = \sum_{j=0}^{\infty} j a_j \xi^{j-1}$$

and

$$\frac{d^2h}{d\xi^2} = 2a_2 + \cdots = \sum_{j=0}^{\infty} (j+1)(j+2) a_{j+2} \xi^j$$

Then

$$\sum_{j=0}^{\infty} \left[(j+1)(j+2) a_{j+2} - 2j a_j + (k-1) a_j \right] \xi^j = 0$$

The coeff.'s vanish (think of the components of a vector $\vec{v} = 0$), so,

$$(j+1)(j+2) a_{j+2} - 2j a_j + (k-1) a_j = 0$$

or

$$a_{j+2} = \frac{(2j+1-k)}{(j+1)(j+2)} a_j$$

At large j we have

$$a_{j+2} \approx \frac{2}{j} a_j \Rightarrow a_j \approx \frac{C}{(j/2)!}$$

So it would seem

$$h(\xi) = C \sum_j \frac{1}{(j/2)!} \xi^j \approx C \sum \frac{1}{j!} \xi^{2j} \approx C e^{\xi^2}$$

But, this blows up!

How can we keep it finite?

We have to terminate the series!! For some n

$$a_{n+2} = 0$$

Then

$$0 = \frac{(2j+1-k)}{(j+1)(j+2)} a_n$$

$$\Rightarrow \boxed{k = 2j+1}$$ quantization!

and so

$$E_n = (n+\tfrac{1}{2})\hbar\omega \quad \text{for } n=0,1,2,\ldots$$

Very cool!

III You already did this for the H-atom

$$u(\rho) = \rho^{l+1} e^{-\rho} v(\rho), \quad \text{where } \rho = \kappa r, \quad \kappa = \frac{\sqrt{-2mE}}{\hbar}$$

you found the recursion relation

$$c_{j+1} = \left\{ \frac{2(j+l+1)-\rho_0}{(j+1)(j+2l+2)} \right\} c_j$$

For large j

$$c_{j+1} \cong \frac{2j}{j(j+1)} c_j = \frac{2}{j+1} c_j$$

$$\Rightarrow c_j = \frac{2^j}{j!} c_0$$

and found v satisfies

$$\rho \frac{d^2 v}{d\rho^2} + 2(l+1-\rho)\frac{dv}{d\rho} + [\rho_0 - 2(l+1)]v = 0$$

Here, $\rho_0 \equiv \frac{me^2}{2\pi\epsilon_0 \hbar^2 \kappa}$.

Again taking the ansatz

$$v(\rho) = \sum_{j=0}^{\infty} c_j \rho^j$$

Then,

$$v(\rho) = c_0 \sum_{j=0}^{\infty} \frac{2^j}{j!} \rho^j = c_0 e^{2\rho}$$

and we get

$$u = c_0 \rho^{l+1} e^{\rho}$$

Not normalizable! Same idea again — we have to truncate the series)

Let j_{max} be the end of the line:

$$2(\underbrace{j_{max} + l + 1}_{\text{integer}}) - \rho_0 = 0$$

Let's give it a shorthand

$$2n = \rho_0$$

$$\Rightarrow E = -\frac{\hbar^2 k^2}{2m} = -\frac{me^4}{8\pi^2 \epsilon_0^2 \hbar^2 \rho_0^2}$$

We conclude,

$$E_n = -\left[\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\right]\frac{1}{n^2}$$

$$= \frac{E_1}{n^2}, \quad n = 1, 2, 3, \ldots$$

$$E_1 = -13.6\,eV.$$

We also see a length scale

$$\frac{1}{nK} = \boxed{a = \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.529 \times 10^{-10}\,m}$$

the Bohr radius, so we have been measuring

$$\rho = \frac{r}{an}$$