# Quantum Mechanics — April 22nd, 2015

Lect 35

I Last time

II $S^2$ for combined systems

III Spin networks part I: pictures as a method to calculate invariants

Using $S_- = S_-^{(1)} + S_-^{(2)}$ we found

$$S_-(\uparrow\uparrow) = \hbar(\downarrow\uparrow + \uparrow\downarrow)$$

and hence that

$$\left.\begin{array}{ll} \overset{s\ m}{|1\ 1\rangle} &= \uparrow\uparrow \\ |1\ 0\rangle &= \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1\ {-1}\rangle &= \downarrow\downarrow \end{array}\right\} \begin{array}{c} s=1 \\ \text{triplet} \end{array}$$

I. Combined behavior of two spin systems

Using $S_z = S_z^{(1)} + S_z^{(2)}$

$\uparrow\uparrow$ : $m=1$

$\uparrow\downarrow$ : $m=0$

$\downarrow\uparrow$ : $m=0$

$\downarrow\downarrow$ : $m=-1$

$$\left\{ |0\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \right\} \begin{array}{c} s=0 \\ \text{singlet} \end{array}$$

- The state space is written

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

Let's confirm that this organization really makes sense.

II Recall that

$$S^2|s,m\rangle = \hbar^2 s(s+1)|s,m\rangle.$$

So we ~~We define~~ expect

$$S^2|1\,0\rangle = \hbar^2\, 1(1+1)|1\,0\rangle = 2\hbar^2 |1\,0\rangle$$

while

$$S^2|0\,0\rangle = \hbar^2(0)(0+1)|0\,0\rangle = 0|0\,0\rangle = 0$$

We define

$$S^2 = \left(\vec{S}^{(1)} + \vec{S}^{(2)}\right)\cdot\left(\vec{S}^{(1)} + \vec{S}^{(2)}\right)$$

$$= \left[S^{(1)}\right]^2 + \left[S^{(2)}\right]^2 + 2\,\vec{S}^{(1)}\cdot\vec{S}^{(2)}$$

So, let's compute,

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)}(\uparrow\downarrow) = (S_x^{(1)}\uparrow)(S_x^{(2)}\downarrow) + (S_y^{(1)}\uparrow)(S_y^{(2)}\downarrow) + (S_z^{(1)}\uparrow)(S_z^{(2)}\downarrow)$$

$$= \left(\frac{\hbar}{2}\downarrow\right)\left(\frac{\hbar}{2}\uparrow\right) + \left(\frac{i\hbar}{2}\downarrow\right)\left(\frac{-i\hbar}{2}\uparrow\right) + \left(\frac{\hbar}{2}\uparrow\right)\left(-\frac{\hbar}{2}\downarrow\right)$$

$$= \frac{\hbar^2}{4}\left[2\downarrow\uparrow - \uparrow\downarrow\right]$$

Similarly,

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)}(\downarrow\uparrow) = \frac{\hbar^2}{4}\left[2\uparrow\downarrow - \downarrow\uparrow\right]$$

Then,

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)}\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}\left(2\downarrow\uparrow - \uparrow\downarrow + 2\uparrow\downarrow - \downarrow\uparrow\right)$$

$$= \frac{\hbar^2}{4}|1\,0\rangle$$

While,

$$\vec{S}^{(1)}\cdot\vec{S}^{(2)}|0\,0\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}\left(2\downarrow\uparrow - \uparrow\downarrow - 2\uparrow\downarrow + \downarrow\uparrow\right)$$

$$= -\frac{3\hbar^2}{4}|0\,0\rangle$$

Putting it all together

$$S^2|1\,0\rangle = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{\hbar^2}{4}\right)|1\,0\rangle = 2\hbar^2|1\,0\rangle$$

awesome

and

$$S^2|0\,0\rangle = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} - 2\,\frac{3\hbar^2}{4}\right)|0\,0\rangle = 0 \;\checkmark\checkmark$$

That does it, we've proved the triplet / singlet structure suggested above.

But, this was the simplest possible ~~way~~ case — combining just two spin ½ systems.

What if we tried to combine two more general spins or angular momenta?

<u>Huge pain in the butt!</u>

But, the results turn out to be awesome it.

I work on an approach to quantum gravity that builds spacetime out of this procedure.

The good news: there's a slick way to do it just by drawing pictures!!! I'll spend the rest of today and next time intro'ing

III Spin networks are a recipe for turning algebra into pictures. Our brains are particularly good at processing visual information — so they can provide a useful tool.

You have met these notations for vectors

$\vec{v}$, $v^i$, $\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$ ← column vector, $v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

Here's a new one

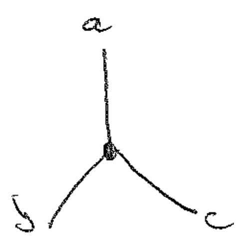
and also

$v_i$, $(v_x, v_y, v_z)$ ← row vector

or

<u>Result</u>: No open wires means the quantity is an (SL) rotational invariant!

This gets more useful when we introduce



$= \epsilon_{abc}$

$= \begin{cases} +1 & \text{cycles of } 123 \\ -1 & \text{cycles of } 213 \end{cases}$, else zero.

is depicted

What aspect of a vector is invariant under rotations? It's length! We can write

$$v^i v_i = v^2 = \vec{v} \cdot \vec{v}$$

You may have encountered

$$\epsilon_{abc}\,\epsilon^{cde}$$

$$= \delta_a^d \delta_b^e - \delta_a^e \delta_b^d$$

This translates to