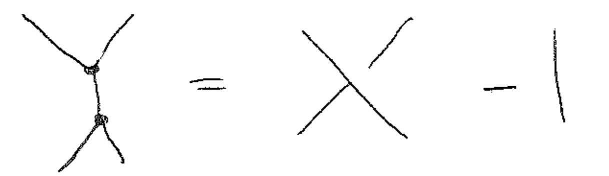
Quantum Mechanics April 29th, 2015 Pl/3

Lect 37

I Last time
II String diagrams for $\mathbb{C}^2$
~~III The case of spinors:~~
~~Spin networks~~

I • Introduced the reduction rule

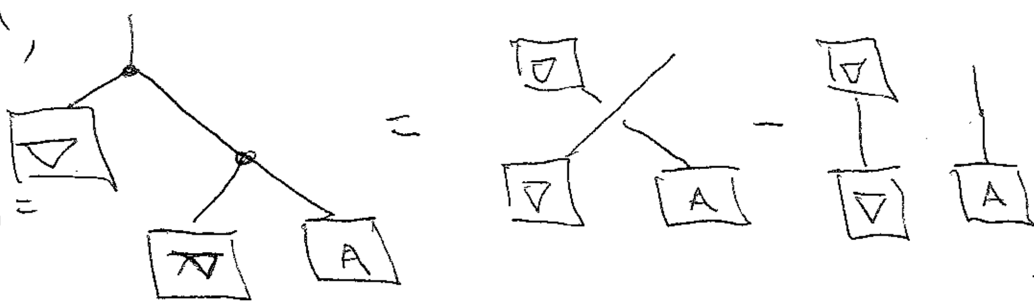
$$\epsilon_{ijk}\,\epsilon^{klm} = \delta_i^{\,l}\,\delta_j^{\,m} - \delta_i^{\,m}\,\delta_j^{\,l}$$



Example: What is $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$?

Well,

$\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) =$



$$= \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} \checkmark$$

Warning: In general a bit tricky to apply these techniques to derivatives because you have to be careful of the product rule.

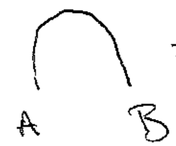
II Recall that a spinor is an element of $\mathbb{C}^2$, e.g.

$$\chi^A = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \quad A = 1, 2.$$

Let's introduce networks for this context,

$= \delta^{A}{}_{B}$ and

a turn instead of a node



$= \epsilon_{AB}$

Recall

$$\epsilon_{AB} = \begin{cases} 0 & A=B \\ 1 & A=1,\ B=2 \\ -1 & A=2,\ B=1 \end{cases}$$

So, $\epsilon_{AB} = -\epsilon_{BA}$

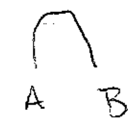
$$\epsilon^{AB}\epsilon_{BC} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\delta^{A}_{C}$$

To get the diagram to match the equation we associate a <u>minus</u> sign to every minimum in a diagram

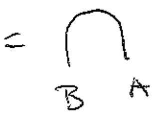
$= -\epsilon^{AB}\epsilon_{BC} = \delta^{A}{}_{C} =$


✓

So we must have



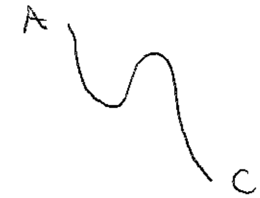
$= \epsilon_{AB}$,


$=$


$= -\epsilon_{AB}$

A minus sign for every crossing

We also want



$=$


but algebraically we have

Hence

$= -\epsilon^{AB}$

<u>Ex</u>: Let's compute

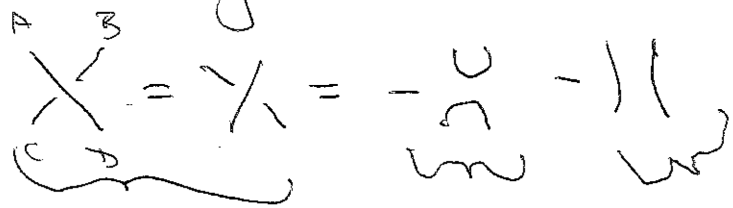

$= -\epsilon^{AB}\epsilon_{AB}$

$= -(1+1) = -2 = -\delta^{A}_{A}$

Once again we have a fundamental <u>reduction rule</u>:

$$\epsilon^{AB}\epsilon_{CD} = \delta^A_C\delta^B_D - \delta^A_D\delta^B_C$$

or in diagrams



$$-\delta^A_D\delta^B_C = \epsilon^{AB}\epsilon_{CD} - \delta^A_C\delta^B_D$$

On the homework you proved that rotations of spinors are given by (Griffiths P 4.56)

$$U = e^{i(\vec{\sigma}\cdot\hat{n})\varphi/2} = \cos(\varphi/2) + i(\hat{n}\cdot\vec{\sigma})\sin(\varphi/2)$$

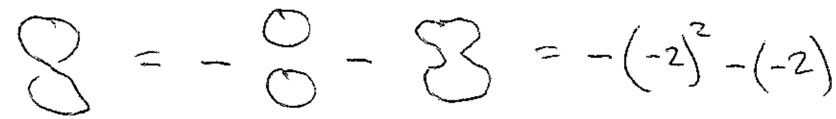
Notice that all the matrices that you get from this satisfy

$$\det U = 1$$

Let's see how this is going

unwinding the string

but also



$$= -2 \checkmark$$

Simple no?

The determinant can be written

$$\det U = \tfrac{1}{2}\,\epsilon^{AC}\epsilon_{BD}\,U_A^B\,U_C^D = 1$$

But this implies

$$U_A^B\,U_C^D\,\epsilon_{BD} = \epsilon_{AC}$$

the epsilon is invariant under rotations!

This means that it is quite useful in characterizing how spins transform under rotations.