Quantum Mechanics — May 1st, 2015 — P1/4

Lect. 38

I Last time

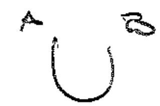
II The case of two spinors

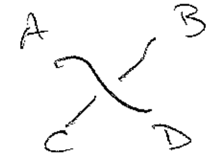
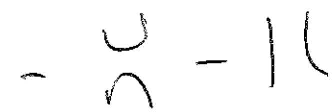
III Tensor product

IV Spinor decomposition ~~and Clebsch's~~.

*Didn't get to this

I. •  $= \delta^A{}_B$  $= \epsilon_{AB}$

 $= -\epsilon^{AB}$

 $= -$ 

$$-\delta^A{}_B\,\delta^B{}_C = \epsilon^{AB}\epsilon_{CD} - \delta^A_C\,\delta^B_D$$

 $= -2$

• Planar homotopy (strings)

RI

RII

• Spinor rotations

$$U = e^{i(\vec{\sigma}\cdot\hat{n})\varphi/2} = \cos(\varphi/2) + i\,(\hat{n}\cdot\vec{\sigma})\sin\left(\tfrac{\varphi}{2}\right)$$

We have

$$\det U = \tfrac{1}{2}\,\epsilon^{AC}\epsilon_{BD}\,U_A^B\,U_C^D = 1$$

and so,

$$U_A^B\,U_C^D\,\epsilon_{BD} = \epsilon_{AC}.$$

~~Spins transform under rotations.~~

III The framework gets richer when we consider composite spin systems.

A completely general state ~~is~~ in $\mathcal{H} = \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$ is ϕ^{AB}, but we can break this into its symmetric and anti-symmetric parts

$$\phi^{AB} = \frac{1}{2}\left(\phi^{AB} - \phi^{BA}\right) + \underbrace{\frac{1}{2}\left(\phi^{AB} + \phi^{BA}\right)}_{\text{call it } \phi^{(AB)}}$$

And Note

$$\phi^{AB} - \phi^{BA} = \delta^A_C \delta^B_D \phi^{CD} - \delta^A_D \delta^B_C \phi^{CD}$$

$$= \epsilon^{AB} \epsilon_{CD} \phi^{CD}$$

So let $\phi = \frac{1}{2} \epsilon_{CD} \phi^{CD}$ and

$$\phi^{AB} = \phi \epsilon^{AB} + \phi^{(AB)}$$

invariant under rotations

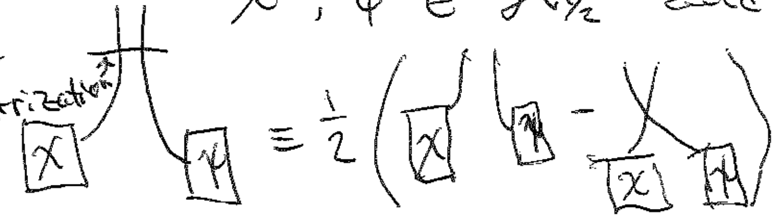
transforms into another symmetric expression under rotations

Should remind you of the singlet/triplet decomposition.

Let's try it. ~~Define~~

Consider $\chi^A, \psi^A \in \mathcal{H}_{1/2}$ ← spin 1/2 and define

Antisymmetrization?



$$= \frac{1}{2}\left(\chi^A \psi^B + \chi^B \psi^A\right)$$

which corresponds to <u>symmetrization</u> of indices.

In a moment we will prove

$$\phi^{(AB)} = \chi^{(A}\psi^{B)}$$

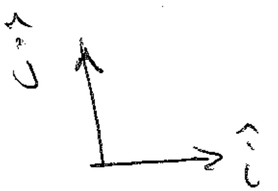
in general. So we see that

$$\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

$$\begin{array}{ccccc} \Downarrow & & \Downarrow & & \Downarrow \\ \phi^{AB} & = & \phi\,\epsilon^{AB} & + & \phi^{(AB)} \end{array}$$

III The tensor product:

$\{\hat{i}, \hat{j}\}$ or



Given ~~$\mathbb{R}$~~ a pair of ~~one~~ vector spaces how can I build a new vector space?

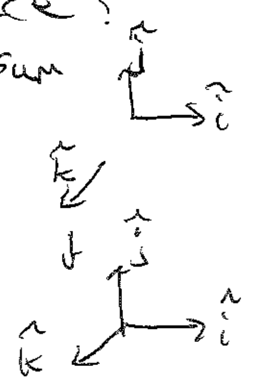
$\mathbb{R}^2$ $\{\hat{i}, \hat{j}\}$

direct sum $\oplus$

$\mathbb{R}^1$ $\{\hat{k}\}$

$\downarrow$

$\mathbb{R}^3$ $\{\hat{i}, \hat{j}, \hat{k}\}$



Before moving on to a general version of this let's finally define $\otimes$.

In practice everything about a (finite-dim.) vector space is determined once you specify a basis.

For example $\mathbb{R}^2$ has basis

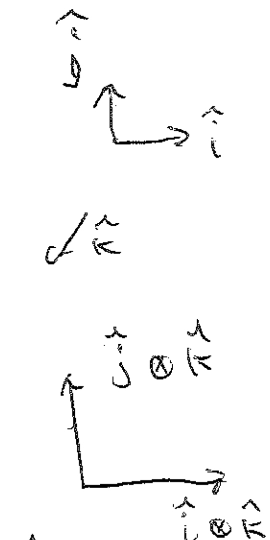
But you can also form the "product" of vector spaces or tensor product

$\mathbb{R}^2$ $\{\hat{i}, \hat{j}\}$ tensor product $\otimes$

$\mathbb{R}^1$ $\{\hat{k}\}$

$\downarrow$

$\mathbb{R}^2$ $\{\hat{i}\otimes\hat{k}, \hat{j}\otimes\hat{k}\}$



Such that $\alpha\hat{i}\otimes\beta\hat{k} = \alpha\beta(\hat{i}\otimes\hat{k})$, $\boxed{\alpha, \beta \in \mathbb{R}}$

Another example,

$\mathbb{C}^2$ with basis $\{\uparrow = \binom{1}{0}, \downarrow = \binom{0}{1}\}$ $\otimes$ $\mathbb{C}^2$ with basis $\{\uparrow = \binom{1}{0}, \downarrow = \binom{0}{1}\}$

$=$ $\mathbb{C}^4$ with basis $\{\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow\}$.

$=$ $\mathbb{C}^4$ with basis $\{\frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow),$

$\uparrow\uparrow, \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), \downarrow\downarrow\}$.

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didn't get to this

IV Let's extend our decomposition to

$$\underset{\chi^A}{\underset{\Downarrow}{\mathbb{C}^2}} \otimes \cdots \otimes \mathbb{C}^2 \otimes \underset{\psi_A}{\underset{\Downarrow}{\mathbb{C}^{2*}}} \otimes \cdots \otimes \mathbb{C}^{2*}$$

star reminds of lower index

How do we get lower indices?

living in here

$$\chi_B \equiv \chi^A \epsilon_{AB}$$

As a warm up, consider $\Phi_{\ldots AB \ldots}$

s.t. $\Phi_{\ldots [AB] \ldots} = \Phi_{\ldots AB \ldots}$

In general

V_1 with basis $\{e_1, e_2, \cdots\}$

V_2 with basis $\{f_1, f_2, \cdots\}$

has

$V_1 \otimes V_2$ with basis

$\{e_1 \otimes f_1, e_1 \otimes f_2, \cdots,$

$e_2 \otimes f_1, e_2 \otimes f_2, \cdots\}$

and $\dim(V_1 \otimes V_2) = \dim V_1 \cdot \dim V_2$

then

$$\epsilon_{AB}\epsilon^{CD}\,\Phi_{\ldots CD \ldots}$$

$$= (\delta_A{}^C \delta_B{}^D - \delta_A{}^D \delta_B{}^C)\,\Phi_{\ldots CD \ldots}$$

$$= 2\,\Phi_{\ldots AB \ldots}$$

and so

$$\Phi_{\ldots [AB] \ldots} = \frac{1}{2}\epsilon_{AB}\,\Phi_{\ldots C}{}^{C}{}_{\ldots}$$

Thus, a general spinor can be written with all the antisymmetr