Quantum Mechanics — Lect. 39

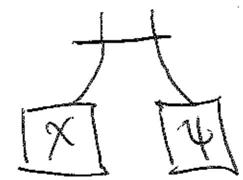
May 4th, 2015

I Last time

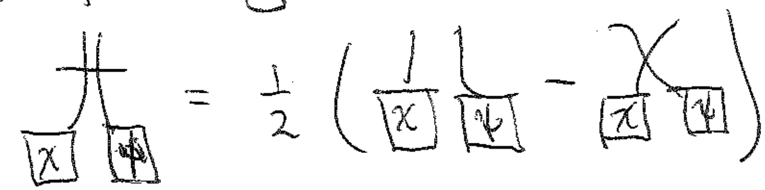
II General spinor decomposition

III Clebsch-Gordan theory

I. What is



graphically?



Algebraically?

$$= \tfrac{1}{2}\left(\chi^A\psi^B + \chi^B\psi^A\right).$$

- Take V_1 with basis $\{e_1, e_2, \cdots\}$

and V_2 with basis $\{f_1, f_2, \cdots\}$

Describe $V_1 \oplus V_2$

$V_1 \oplus V_2$ has basis $\{e_1, e_2, \ldots, f_1, f_2 \cdots\}$

Its vectors can always be writte $\vec{v} \in V_1 \oplus V_2$

$$\vec{v} = \vec{v}_1 + \vec{v}_2.$$

Also, $\dim V_1 \oplus V_2 = \dim V_1 + \dim V_2$

Describe $V_1 \otimes V_2$:

it has basis $\{e_1\otimes f_1, e_1 \otimes f_2, \ldots, e_2\otimes f_1, e_2 \otimes f_2, \cdots\}$.

$$\dim V_1 \otimes V_2 = (\dim V_1)(\dim V_2).$$

II Let's extend our decomposition

$$\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} = \mathcal{H}_0 \oplus \mathcal{H}_1$$

to

$$\mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} \otimes \cdots \otimes \mathcal{H}^*_{1/2} \otimes \cdots \otimes \mathcal{H}^*_{1/2}$$

or equivalently

$$\underset{\substack{\cup\\ \chi^A}}{\mathbb{C}^2} \otimes \mathbb{C}^2 \otimes \cdots \otimes \underset{\substack{\psi\\ A}}{\mathbb{C}^{2*}} \otimes \cdots \otimes \mathbb{C}^{2*}$$

living in here

Star reminds us of lower index

$$\epsilon_{AB}\,\epsilon^{CD}\,\Phi_{\dots CD \dots} = (\delta^C_A \delta^D_B - \delta^D_A \delta^C_B)\Phi_{\dots CD \dots}$$

$$= 2\,\Phi_{\dots AB \dots}$$

and so

$$\Phi_{\dots [AB] \dots} = \frac{1}{2}\epsilon_{AB}\,\Phi_{\dots C}{}^{C}{}_{\dots}$$

This means that whenever a general spinor has an antisymmetric pair of indices, we can explicitly factor that antisymmetry out — expressing it explicitly in terms of an ϵ.

How do we get lower indices?

$$\chi_B \equiv \chi^A \epsilon_{AB}$$

As a warm up, consider $\Phi_{\dots AB \dots}$ such that

$$\Phi_{\dots AB \dots} = \Phi_{\dots [AB] \dots}$$

then

~~explicit in ϵ's~~.

So, now let's consider symmetric groups of indices

$$\underbrace{\Theta_{A\cdots B}}_{K \text{ indices}} = \Theta_{(A\cdots B)} \equiv \frac{1}{K!}\sum_{\pi} T_{i_{\pi 1} i_{\pi 2}\cdots}$$

If we contract this with a set of vectors $\xi^A = (z, 1)$ we get

$$\Theta_{A\cdots B}\,\xi^A \cdots \xi^B = \underset{\text{in } z}{\text{Polynomial}} \equiv P(z)$$

But this means

$$U^A{}_L \cdots U^B{}_M\, \Theta_{A\cdots B}$$

$$= (U^A{}_L\, \chi_A)\cdots(U^B{}_M\, \psi_B) \quad \text{still symmetrical}$$

$$= \Theta'_{A\cdots B} = \Theta'_{(A\cdots B)}$$

These symmetrical ~~groups~~ types transform into themselves!

But, we're working over the complex numbers and so every such polynomial factorizes into linear factors (fundamental theorem of alg.)

$$P(z) = \underbrace{(b_1 z + b_2)}\,\underbrace{(c_1 z + c_2)}\cdots\underbrace{(h_1 z + h_2)}$$

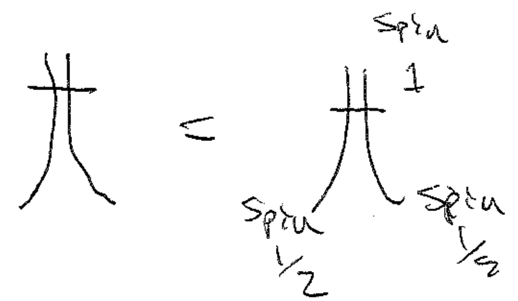
$$= \chi_A\,\xi^A \cdots \psi_B\,\xi^B$$

So $\Theta_{A\cdots B} = \chi_{(A}\cdots\psi_{B)}$!

Putting together our results on symmetry and antisymmetry provides a powerful classification of spinors.

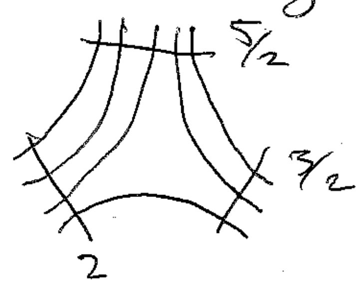
III Pulling out any antisymmetry as ϵ's the irreducible parts of any object $\Psi_{A\cdots B}$ are symmetric

The core of our graphical example was



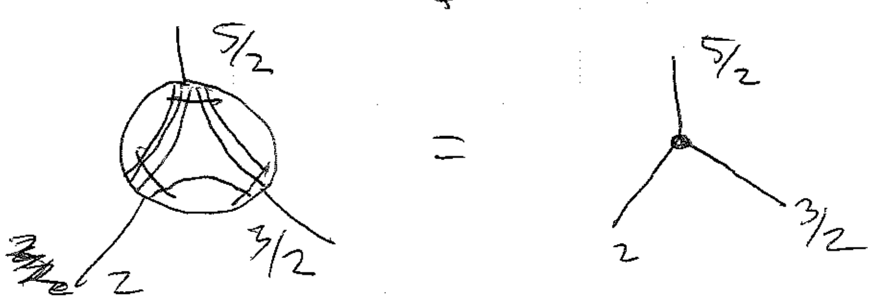
this diagram represents a tensor that can be used to couple two spin 1/2 particles to yield the spin 1, state $\frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow)$.

extends to any number of strings



(5 spin 1/2's added to give spin 5/2)

This is often abbreviated to



We have just shown that anti-sym of strings corresponds to sym. of indices and that the index symmetric subspaces carry the irreducible subspaces under SU(2) rotations.

So, the basic strategy

What is the general result? If we couple spin j_1 and j_2 what are the possible outcomes of coupling?

In the string diagram language this is a routing of $a = 2j_1$ and $b = 2j_2$ strands through a node resulting in a rope with $c = 2j_3$ strands, a, b and c all integers.

If no a-strands connect to b-ones then $c = a+b$ and $j_3 = j_1 + j_2$.
If 1 a-strand connects to a b-strand then $c = a+b-2$ and $j_3 = j_1 + j_2 - 1$.

Because every string has two ends $a+b+c$ is even and $\boxed{j_1+j_2+j_3 \text{ is integer!}}$

So it is possible to obtain j_3 in coupling j_1 and j_2 if

$$|j_1 - j_2| \le j_3 \le j_1 + j_2 \quad \text{and} \quad j_1 + j_2 + j_3 = n$$

with n an integer.

Continuing to connect strands one at a time j_3 decreases in integer steps until all possible ab-strand connections are made. The remaining strands give

$$c = |a-b| \Rightarrow j_3 = |j_1 - j_2|$$