Quantum Mechanics

May 6th, 2015

Lect. 40

I Diagramatic outro

II Two particles & ~~identical particles~~

III Identical particles: Bosons and fermions

I. • I hope you enjoyed our brief tour of diagrammatic techniques and tensors. These tools are useful <u>throughout</u> your entire study of physics — keep them in mind when you feel you've hit a systematic barrier!

• Our (beautiful) result that concluded last time is: When you couple j_1 and j_2 to get j_3 the possible results are

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2$$

and

$$j_1 + j_2 + j_3 = n$$

with n an integer.

II Recently we have tackled the problem of two spins. How does their spins combine? What is their state space?

~~What~~ This leads to similar questions for two particles: How do we describe the wave function of a system consisting of two particles?

Take $\vec{r}_1$ to describe the position of the 1st particle and $\vec{r}_2$ that of the 2nd.

Then

$$\Psi(\vec{r}_1, \vec{r}_2, t)$$

is the wavefunction.

It solves

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi$$

where

$$H = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2, t).$$

(Recall $\vec{\nabla}_1 = \vec{\nabla}_{\vec{r}_1} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial z_1}\right)$.)

We have

$$|\Psi(\vec{r}_1, \vec{r}_2, t)|^2 d^3\vec{r}_1 d^3\vec{r}_2 =$$ Prob of finding particle 1 in vol $d^3\vec{r}_1$ and particle 2 in vol $d^3\vec{r}_2$.

and

$$\int |\Psi|^2 d^3\vec{r}_1 d^3\vec{r}_2 = 1$$

For time indep. potentials $V(\vec{r}_1, \vec{r}_2, t) = V(\vec{r}_1, \vec{r}_2)$ we have

$$\Psi(\vec{r}_1, \vec{r}_2, t) = \psi(\vec{r}_1, \vec{r}_2) e^{-\frac{iEt}{\hbar}}$$

and

$$-\frac{\hbar^2}{2m_1}\nabla_1^2\psi - \frac{\hbar^2}{2m_2}\nabla_2^2\psi + V\psi = E\psi.$$

III Identical particles: Bosons and Fermions:

Every electron ever observed has exactly the same mass, charge, and spin.

If you think about this it's really quite incredible. How?

Ignore spin for a moment and consider two distinguishable particles! Suppose that particle 1 has state $\psi_a(\vec{r})$ and particle 2 has $\psi_b(\vec{r})$ then we can write

Note that this is a very special state.

Distinguishable: $\psi(\vec{r}_1, \vec{r}_2) = \psi_a(\vec{r}_1)\,\psi_b(\vec{r}_2)$

Griffiths says of indisting. particles*: "It's not just that we don't happen to know which electron is which; God doesn't know which is which ..."

Quantum mechanics has a neat way of incorporating this; there are two possibilities

$$\psi_\pm(\vec{r}_1, \vec{r}_2) = A\left[\psi_a(\vec{r}_1)\,\psi_b(\vec{r}_2) \pm \psi_b(\vec{r}_1)\,\psi_a(\vec{r}_2)\right]$$

Bosons (+)

Fermions (−)

The $\pm$ defines bosons and fermions.

There is a stunning theorem of quantum field theory:

all particles with integer spin are bosons, and all particles with half integer spin are fermions

The spin and statistics theorem.

A wonderful consequence is that <u>two identical fermions cannot occupy the same state</u> because

$$\psi_-(\vec{r}_1,\vec{r}_2) = A[\psi_a(\vec{r}_1)\psi_a(\vec{r}_2) - \psi_a(\vec{r}_1)\psi_a(\vec{r}_2)] = 0!$$

This is the <u>Pauli exclusion principle</u>

In fact, the case we have been considering is a special case of a more

eigenvalues are $\lambda_\pm = \pm 1$.

If the two particles are identical $m_1 = m_2$ and $V(\vec{r}_1,\vec{r}_2) = V(\vec{r}_2,\vec{r}_1)$ and so P and H satisfy

$$[P, H] = 0$$

and are compatible observables.

Thus we can find solutions s.t.

general property.

Define the <u>exchange operator</u> P by

$$P f(\vec{r}_1,\vec{r}_2) = f(\vec{r}_2,\vec{r}_1).$$

We have,

$$P^2 = 1$$

and you will show on HW that its two

$$\psi(\vec{r}_1,\vec{r}_2) = \pm\psi(\vec{r}_2,\vec{r}_1). \quad (*)$$

If a state begins its life in such a state, the evolution doesn't disturb this property.

We have a new law, the symmetrization requirement, is that for identical particles ψ is <u>required</u> to satisfy $(*)$; $+$ = bosons, $-$ = fermions

<u>Example</u>: Two particles in ∞-square well.

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right), \quad E_n = n^2 K$$

with $K = \dfrac{\pi^2 \hbar^2}{2ma^2}$

<u>Distinguishable</u>:

$$\psi_{n_1 n_2}(x_1, x_2) = \psi_{n_1}(x_1)\,\psi_{n_2}(x_2), \quad E_{n_1 n_2} = (n_1^2 + n_2^2)K$$

$$\psi_{\substack{\text{1st} \\ \text{excited}}} = \frac{\sqrt{2}}{a}\left[s\left(\frac{\pi x_1}{a}\right) s\left(\frac{2\pi x_2}{a}\right) + s\left(\frac{2\pi x_1}{a}\right) s\left(\frac{\pi x_2}{a}\right)\right]$$

still with $E = 5K$, but no degeneracy!

For fermions $\rightarrow$ the ground state from above disappears!

$$\psi_{\text{grnd}} = \frac{\sqrt{2}}{a}\left[s\left(\frac{\pi x_1}{a}\right) s\left(\frac{2\pi x_2}{a}\right) - s\left(\frac{2\pi x_1}{a}\right) s\left(\frac{\pi x_2}{a}\right)\right]$$

with <u>ground state</u> $E = 5K$!

So, the grnd state

$$\psi_{11} = \frac{2}{a} s\left(\frac{\pi x_1}{a}\right) s\left(\frac{\pi x_2}{a}\right), \quad E_{11} = 2K$$

while

$$\psi_{12} = \frac{2}{a} s\left(\frac{\pi x_1}{a}\right) s\left(\frac{2\pi x_2}{\cancel{2}a}\right), \quad E_{12} = 5K$$

$$\psi_{21} = \frac{2}{a} s\left(\frac{2\pi x_1}{a}\right) s\left(\frac{\pi x_2}{a}\right), \quad E_{21} = 5K.$$

For bosons ψ_{11} is same as above soln,