Quantum Mechanics
Lect. 6

Feb. 6th, 2015

I Last time

II The opto-mechanical analogy

III Schrödinger's insight

I
- Applied Fermat's principle to derive Snell's law
- Cast Fermat's principle in the general form:

$$T = \cancel{\frac{1}{c}} \int_S^D \frac{n(s)}{c}\, ds$$

is an extremum

- ~~Exp~~ Huygens' principle: Successive wave fronts are due to the superposition of spherical wavelets sourced on the present wavefront

Fermat's principle follows naturally from this construction.

II Hamilton noticed a striking correspondence. If we think of mechanics as the pursuit of finding trajectories then it is quite similar to finding optical rays.

If energy is conserved what determines how quickly a particle moves through space?

Well, $$\frac{p^2}{2m} + V = E$$

So $$p = \sqrt{2m(E-V)}.$$

We see that essentially the potential energy acts like an index of refraction of the particle.

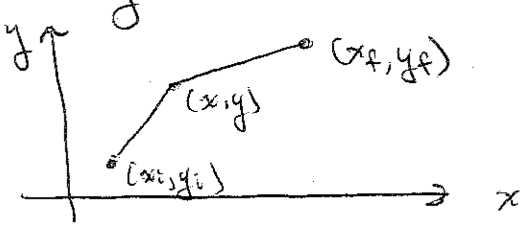
This leads him to introduce

Note that

$$[S] = [p][q]$$

= momentum · position
= angular momentum ($\vec{L} = \vec{r} \times \vec{p}$)
= action
= Energy · time ($\frac{1}{2}pv \cdot t$)

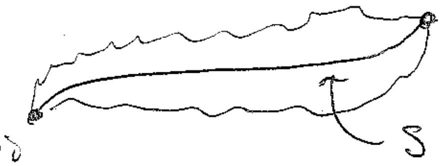
Let's try an example: A free particle



Hamilton's principle:

A particle travels the path between two fixed points in space that extremizes the <u>action</u>

$$S = \int_{q_i}^{q_f} p \, dq$$



Then

$$S_1 = \sqrt{2mE}\,\overbrace{\sqrt{(x-x_i)^2 + (y-y_i)^2}}^{\equiv s_1}$$

$$S_2 = \sqrt{2mE}\,\overbrace{\sqrt{(x_f-x)^2 + (y_f-y)^2}}^{\equiv s_2}$$

and

$$S = S_1 + S_2$$

We extremize, first x,

$$\frac{\partial S}{\partial x} = 0 = \frac{(x-x_i)}{s_1} - \frac{(x_f-x)}{s_2}$$

$$\frac{\partial S}{\partial y} = 0 = \frac{(y-y_i)}{s_1} - \frac{(y_f-y)}{s_2}$$

$$\Rightarrow \quad \frac{x-x_i}{x_f-x} = \frac{y-y_i}{y_f-y}$$

or

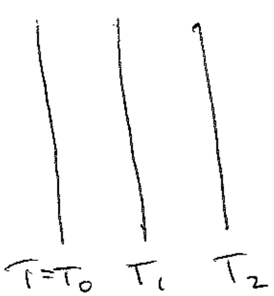
$$\frac{y_f-y}{x_f-x} = \frac{y-y_i}{x-x_i}$$

The slopes are equal! It should be a straight line.

This leads us to ask whether a wave explanation can also be given for Hamilton's principle?

III Schrodinger's insight was to see if he could build a wave theory for particle mechanics.

Starting from a wavefront $T_0 = \text{const.}$ we reach the next wave front by following the wavelets for a fixed length of time. Hence, all the level sets of the wave field are give by surfaces of constant T.



or

or any intermediate height too.

So, let's require the same thing for S, that is,

$$\frac{1}{\hbar} S(x+\lambda,t) = \frac{1}{\hbar} S(x,t) + 2\pi$$

Necessary for dimensional consistency — we could call it c for constant and interpret later if we wanted.

Now, suppose λ is small, then we can Taylor expand and find

$$\frac{1}{\hbar}\left(S(x,t) + \lambda \frac{\partial S}{\partial x} + \dots\right) = \frac{1}{\hbar} S(x,t) + 2\pi$$

Simplifying this is,

$$\frac{\partial S}{\partial x} = \frac{2\pi\hbar}{\lambda}$$

but

$$S = \int p\, dx \qquad \text{recall } q = \text{position} = x$$

so

$$\boxed{\frac{\partial S}{\partial x} = p = \frac{h}{\lambda}}$$

de Broglie's hypothesis comes right out of the wave idea! This provides a compelling impetus to the wave idea.

Next week we will start to build this wave theory towards Schrödinger's eqn.