Quantum Mechanics Lect. 7

0. Questions about the guest lecturing?

I Last time

II The Schrödinger Equation

III Functions as vectors?

I. • Introduced the classical action

$$S = \int_{q_i}^{q_f} p\, dq$$

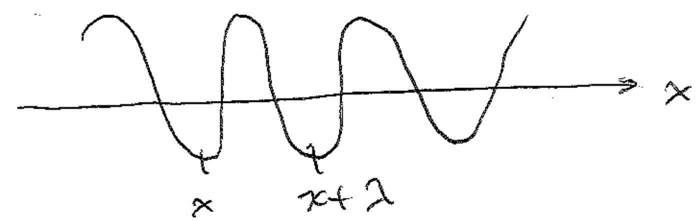
or Hamilton's full version

$$S = \int_{q_i}^{q_f} p \underbrace{\dot{q}\, dt}_{dq} - H\, dt$$

Surfaces $H(q,p,t) = \text{const} \equiv E$ are constant energy surfaces

• Hamilton's principle: the physical trajectories connecting q_i and q_f are extremals of the action.

• Schrödinger built a wave theory of particles out of this observation



$$\frac{1}{\hbar} S(x+\lambda, t) = \frac{1}{\hbar} S(x,t) + 2\pi$$

But, by Taylor expansion

$$\frac{1}{\hbar}\frac{\partial S}{\partial x}\lambda = 2\pi \Rightarrow \boxed{p = \frac{\partial S}{\partial x} = \frac{h}{\lambda}}$$

Each wavefront can be

thought of as a swarm of particles (like Huygens' wavelets) that follow classical trajectories.

II For the electric field

$$\vec{E}_{pl} = \vec{E}_0 \, e^{i(kx - \omega t)}$$

plane wave → ; related to momentum ; related to energy.

So let's consider Hamilton's full principle with

or

$$\frac{\partial S}{\partial t} = -\frac{2\pi\hbar}{T} = -\hbar\omega$$

But from the action

$$\frac{\partial S}{\partial t} = -E$$ ← constant level of H.

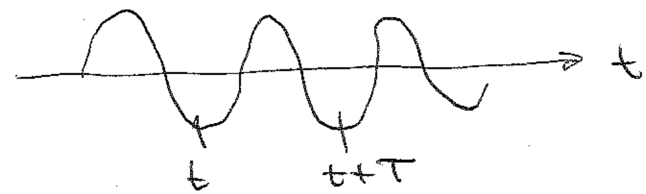
and so,

$$E = \hbar\omega$$ (Einstein relation).

Now, note that we can always

$$S = \int_{q_i}^{q_f} (p\dot{q} - H)\, dt$$

Well then let's try it again



$$\frac{1}{\hbar} S(x, t+T) = \frac{1}{\hbar} S(x,t) - 2\pi$$

Taylor expanding

$$\frac{1}{\hbar}\frac{\partial S}{\partial t} T = -2\pi$$

break a wave up into a superposition of waves of the form

$$\vec{E}_p = \vec{E}_0 \, e^{i(kx - \omega t)}$$

Let's do the same with our matter wave

$$\Psi_{pl}(x,t) = R \, e^{\frac{i}{\hbar} S}$$ ← action

plane wave → ; amplitude

Let's take a standard form for the energy

$$H(x, p, t) = \frac{p^2}{2m} + V(x,t)$$

and note

$$\frac{\hbar}{i}\frac{\partial}{\partial x}\psi_{P\ell} = \frac{\hbar}{i} R e^{i/\hbar S} \cdot \frac{i}{\hbar}\frac{\partial S}{\partial x}$$

$$= P \cdot \psi_{P\ell}$$

Similarly $\frac{\hbar}{i}\frac{\partial}{\partial t}\psi_{P\ell} = \frac{\hbar}{i}\frac{i}{\hbar}\frac{\partial S}{\partial t} R e^{i/\hbar S} = -E\,\psi_{P\ell}$

or $\boxed{i\hbar \frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x,t)\,\psi}$

Schrödinger's equation.

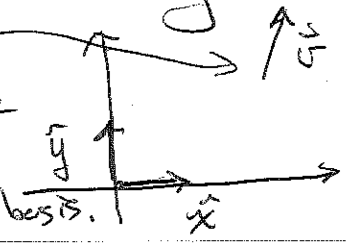
I think this is a beautiful piece of theoretical physics — he drew together several strands and used an inductive leap to arrive at the correct generalization.

Putting these observations together (Fermat, Huygens, Hamilton) he ~~so inducted~~ ~~used induction to~~ guessed that any quantum wave should satisfy

$$-\frac{\hbar}{i}\frac{\partial}{\partial t}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi$$

III I don't need to convince you of the utility of the idea of a vector — you've seen it. You also know the power of choosing a basis

Can decompose any $\vec{v}$ in $\{\hat{x}, \hat{y}\}$ basis.



$\mathbb{R}^2$ as a vector space

But, have you noticed that you can do the exact same thing with functions $f(x)$?

Suppose $f(x)$ is periodic with period 2π then we can write

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

in this basis.

Let's do some examples.

On Wed. Adrian is going to be explaining the general conditions you need on a set of functions for them to act like a basis of functions.

$\cos(0x)$ $\quad\sin(x)$

$\cos(x)$ $\quad\sin(2x)$

$\cos(2x)$ $\quad\sin(3x)$

$\vdots \qquad \vdots$

act like basis "vectors"!

a_0, a_n and b_n act like components of ~~these~~ function $f(x)$

Check it out!

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \pi\,\delta_{mn}$$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \pi\,\delta_{mn}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0 \quad \forall\, m,n.$$