Quantum Mechanics
Day 10

Today
I. Last Time & Exam Reminder
II. Determinate States
III. Observable Determinate States (or Eigenstates of a Hermitian Operator)

I. Last Time

- We defined hermitian operators as those that satisfy

$$\langle f|\hat{Q}f\rangle = \langle \hat{Q}f|f\rangle \quad \text{for all } f(x).$$

- We defined the adjoint operator $\hat{Q}^\dagger$ by

$$\langle \hat{Q}^\dagger f|g\rangle = \langle f|\hat{Q}g\rangle.$$

  Griffiths and Schroeter also call this the hermitian conjugate.

- We argued that

  > Observables are represented by hermitian operators.

  We also gave a useful characterization of hermiticity; an operator $\hat{Q}$ is hermitian if $\hat{Q}^\dagger = \hat{Q}$.

- We claimed that eigenvalues and eigenvectors were special. The first giving measurement outcomes and the second being special states, whose measurement resulted in definite values. Can we argue for this claim?

- We plan to have an in-class exam on Monday, March 2nd.

II. Determinate States

First let us define an Ensemble = {Huge # of identically prepared states}. I like to think of a Hollywood squares of quantum states. If you measure the upper left state you get one measurement outcome, and if you measure the state in the 2nd row and 3rd column you get a different measurement outcome in general. As we have discussed, this means that we have to describe measurements statistically!

But, are there some states that have definite outcomes, say when you measure $\hat{Q}$? That is, are there determinate states of $\hat{Q}$? Suppose there were, then certainly $\sigma^2 = 0$ for these states. We have

$$\begin{aligned} 0 = \sigma^2 &= \langle (\hat{Q} - \langle \hat{Q}\rangle)^2\rangle \\ &= \langle \Psi|(\hat{Q} - \langle \hat{Q}\rangle)^2\Psi\rangle \\ &= \langle \Psi|(\hat{Q} - q)^2\Psi\rangle \\ &= \langle (\hat{Q} - q)\Psi|(\hat{Q} - q)\Psi\rangle, \end{aligned}$$

where in the 3rd equality I've used the fact that the expectation value of a determinate state would have to be the value of $\hat{Q}$ on that state and in the 4th the hermiticity of the observable. But, the only state that is orthogonal to itself is the vanishing state, so

$$|(\hat{Q}-q)\Psi\rangle = 0 \qquad \Longrightarrow \qquad \boxed{\hat{Q}\Psi = q\Psi.}$$

Thus hermiticity and the determinate nature of a state completely determine the fact that the state must be an eigenstate of the operator.

III. Observable Determinate States (or Eigenstates of an Hermitian Operator)

What further specification can we make on determinate states that also happen to be eigenstates of an hermitian operator? There are two important cases: when the measurement outcomes are discrete, we say that the operator $\hat{Q}$ has a 'discrete spectrum' $\{q\}_{i=1}^{N}$; when the measurement outcomes are in a continuous interval or are the whole real line, we call it a 'continuous spectrum'.

Discrete Spectrum: Let's focus on the discrete spectrum for the moment. Then we have two theorems on hermitian operators.

Theorem 1: The eigenvalues of a hermitian operator with discrete spectrum are real.

Proof: Suppose $\hat{Q}f = qf$ and $\langle \hat{Q}f|f\rangle = \langle f|\hat{Q}f\rangle$, then

$$\langle qf|f\rangle = \langle f|qf\rangle \quad \text{or} \quad q^*\langle f|f\rangle = q\langle f|f\rangle.$$

Since f can't vanish, it follows that $q^* = q$, i.e. that q is real.

Theorem 2: The eigenstates for distinct q are orthogonal.

Proof: Suppose $\hat{Q}f = qf$ and $\hat{Q}g = q'g$, then

$$\langle f|\hat{Q}g\rangle = \langle \hat{Q}f|g\rangle \qquad \Longrightarrow \qquad q'\langle f|g\rangle = q^*\langle f|g\rangle,$$

but by assumption q and q' are distinct, so, $q \neq q'$ implies that

$$\boxed{\langle f|g\rangle = 0.}$$

Note that this theorem tells us nothing about distinct eigenstates, say f and g, that happen to have the same q. However, the Gram-Schmidt procedure allows you to construct an orthonormal basis amongst all such eigenfunctions.

Theorem 3: If $\dim \mathscr{H}$ = a finite #, then the eigenvectors of any hermitian $\hat{Q}$ span all of $\mathscr{H}$.

There is no such theorem in the infinite dimensional context—but, this is required for quantum theory. So, along with Dirac we take it as an axiom: The eigenfunctions of an observable operator are complete.

Continuous Spectrum: The eigenfunctions in this case are not normalizable! Nonetheless we can proceed: Ex. What are the eigenfunctons of $\hat{p}$? Call them f_p. We've already solved this, recall Ψ_{plane}, but let's make our solution more explicit using the eigenfunction condition

$$\frac{\hbar}{i}\frac{d}{dx}f_p(x) = pf_p(x).$$

Treating this as a differential equation, the general solution is

$$f_p(x) = Ae^{ipx/\hbar}.$$

Wonderfully this agrees with what we were saying before (replacing R by A to make notations match)

$$\Psi_{\text{plane}} = Ae^{iS/\hbar},$$

where $S = \int pdx$ and for a free particle momentum is conserved and so

$$S = p\int dx = px.$$

Hence

$$\Psi_{\text{plane}} = Ae^{ipx/\hbar}.$$

An odd property of these solutions is that they are not normalizable, check this yourself, and hence don't live in the Hilbert space. Nonetheless, they do satisfy something like orthonormality

$$\begin{aligned}\int_{-\infty}^{\infty} f_{p'}^*(x)f_p(x)dx &= |A|^2\int_{-\infty}^{\infty} e^{i(p-p')x/\hbar} \\ &= |A|^2 2\pi\hbar\delta(p-p'),\end{aligned}$$

here $\delta(p-p')$ is called the Dirac delta function and is defined by

$$\delta(p-p') = \begin{cases}\infty & \text{if } p = p' \\ 0 & \text{if } p \neq p'\end{cases}$$

or, more generally, by

$$\delta(x) = \begin{cases}\infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0,\end{cases}$$

and such that

$$\int \delta(x)dx = 1.$$

Now if we take $A = 1/\sqrt{2\pi\hbar}$ we have

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$$

and

$$\langle f_{p'}|f_p\rangle = \delta(p-p').$$

This is now awfully similar to orthonormality—it's a sort of continuum version that we call Dirac orthonormality.

Even better the f_p are complete:

$$f(x) = \int_{-\infty}^{\infty} c(p) f_p(x) dp = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p) e^{ipx/\hbar} dp$$

and

$$\langle f_{p'} | f \rangle = c(p').$$

We will explore all of this in more detail later in the semester. For the moment I just want you to have a preview of it and to start to be acquainted with its oddities.

Notice the pleasant convergence of our ideas

$$f_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

is a plane wave with

$$\boxed{\lambda = \frac{2\pi\hbar}{p} = \frac{h}{p}.}$$

We have a new appreciation for these subtle eigenfunctions $\rightsquigarrow$ they're not in the Hilbert space.

If the spectrum of a hermitian $\hat{Q}$ is continuous, the eigenfunctions are not normalizable and don't live in $\mathscr{H}$. Nonetheless, they are Dirac orthonormalizable and complete.