Quantum Mechanics
Day 11

Today
I. Last Time & Extension Discussion
II. Time-Independent Schrödinger Equation & Stationary States
III. Julia's Guest Lecture: The Infinite Square Well

I. Last Time

- We took our first look at observable determinate states
  Discrete Spectra: – eigenvalues of hermitian $\hat{Q}$ are real
  – eigenfunctions for distinct eigenvalues are orthogonal
  – If dim $\mathscr{H}$ =finite #, then eigenvectors of $\hat{Q}$ span $\mathscr{H}$.
  $\rightsquigarrow$ take as an axiom in the continuous case.

- Found the eigenfunctions of $\hat{p}$ to illustrate **continuous spectrum** case
$$f_p = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}.$$
  Not normalizable, but Dirac orthonormal
$$\langle f_{p'} | f_p \rangle = \delta(p - p').$$
  They are complete! That is, for some $c(p)$ we can construct:
$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} c(p) f_p(x)\, dp \\ &= \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} c(p) e^{ipx/\hbar}\, dp, \end{aligned}$$
  and, as before,
$$\langle f_{p'} | f(x) \rangle = c(p').$$
  We'll show this soon.

- Thus, if the spectrum of a hermitian $\hat{Q}$ is continuous, the eigenfunctions are not normalizable, and don't live in $\mathscr{H}$. Nonetheless, they are Dirac orthonormalizable and complete.

- Because of the way that we have structured the course it is very difficult for me to give extensions on an individual basis. I will not be doing this. However, I'm happy to discuss whether our homework deadline is working.

II. Time-Independent Schrödinger Equation & Stationary States

Let's return to the dynamics of Quantum Theory, that is, to the Schrödinger equation
$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi.$$

We will assume that $V(x,t) = V(x)$, that is, that the potential has no t dependence. Then we can use separation of variables to solve this partial differential equation (PDE). This is an extremely general technique that is useful for almost all of the PDEs that you will encounter in physics and an essential technique to master. The idea is to assume that it is possible to separate

$$\Psi(x,t) = \psi(x)\varphi(t).$$

Notice the different meanings of the capital and lower case psi letters. When this is possible, the Schrödinger equation becomes

$$i\hbar\psi\frac{d\varphi}{dt} = -\frac{\hbar^2}{2m}\varphi\frac{d^2\psi}{dx^2} + V\psi\varphi$$

and if we divide through by $\psi\varphi$ we get

$$\underbrace{i\hbar\frac{1}{\varphi}\frac{d\varphi}{dt}}_{\substack{\text{only a}\\ \text{function of } t}} = \underbrace{-\frac{\hbar^2}{2m}\frac{1}{\psi}\frac{d^2\psi}{dx^2} + V}_{\substack{\text{only a function of } x\\ \text{(here's where}\\ V=V(x) \text{ comes in)}}}.$$

When can $f(t) = g(x)$? This is only possible if $f = g = \text{const.} \equiv E$. Then

$$\boxed{\frac{d\varphi}{dt} = -\frac{i}{\hbar}E\varphi,}$$

and

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = E\psi.}$$

This is the time-independent Schrödinger equation. We can solve the first of these equations for φ once and for all

$$\varphi = e^{-\frac{i}{\hbar}Et}$$

(we'll put the overall constant into our solution for ψ). But, these separable states are only special solutions of the Schrödinger equation. Who cares about them?! There are at least three reasons to care—we'll cover one now (more to follow soon):

1. They have definite total energy. The classical energy is

   $$H(x,p) = \frac{p^2}{2m} + V(x)$$

   and leads us to introduce the Hamiltonian operator

   $$\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x).$$

   We find that ψ is a determinate state with

   $$\hat{H}\psi = E\psi,$$

   in fact, this is a way of writing the time independent Schrödinger equation, since $\psi = \psi(x)$.

III. Julia's Guest Lecture: The Infinite Square Well

I will scan and add my notes from Julia's lectures as soon as I'm able.