Quantum Mechanics
Day 13

Today
I. Last Time
II. Ubiquity of Harmonic Oscillators
III. The Algebraic, Ladder Operator Method
IV. The Prestige

I. Last Time

- Square wells, although quite idealized, are useful in applications, e.g. the nuclear shell model. Protons and neutrons are tightly bound in the nucleus due to the strong force—the potential can be seen as a <u>finite</u> square well. We'll study this soon. Also used in optoelectronics, quantum well lasers, quantum well infrared detector, et alius.
- We found a general state from its initial condition

$$\Psi(x,0) = \sum_{n=1}^{\infty} c_n \psi_n(x)$$

  with

$$c_n = \langle \psi_n | \Psi(x,0) \rangle = \int \psi_n^* \Psi(x,0) dx$$

  and then

$$\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n e^{-iEt/\hbar}.$$

- We applied this to one example initial state and saw how we could animate the result using Python. This led us to numerically discover the phenomenon of wave function revival.

II. Ubiquity of Harmonic Oscillators

We saw this already formally when we reviewed the Taylor series, but let's go over it again in a physical way. The famous force formula for a harmonic oscillator

$$F = -kx = m\frac{d^2x}{dt^2}$$

has solution

$$x(t) = A\sin(\omega t) + B\cos(\omega t),$$

with $\omega \equiv \sqrt{k/m}$. Viewed as a potential problem, this is

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2.$$

The reason this potential is so important is that it resembles <u>any</u> potential near its local minima. To confirm this we again Taylor expand

$$V(x) = \underbrace{V(x_0)}_{\text{just a shift}} + \cancelto{0}{V'(x_0)}(x-x_0) + \frac{1}{2}V''(x_0)(x-x_0)^2 + \cdots.$$

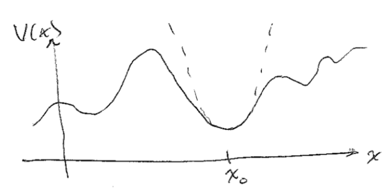

Figure 1: The local structure of an arbitrary potential in the neighborhood of one of its minima.

So,

$$V(x) \approx \frac{1}{2}V''(x_0)(x-x_0)^2 = \frac{1}{2}k(x-x_0)^2,$$

with $k = m\omega^2 = V''(x_0)$.

The ubiquity of this potential leads us to tackle the quantum problem

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi.$$

The non-linear dependence on the independent variable in the x^2 term makes this ODE tricky to solve. We'll use an unfamiliar, but amazing, algebraic method called the 'ladder operator' method today. In time we will also tackle this directly as a differential equation. While the ladder operator method is special, it will recur again and again in your study of quantum mechanics and quantum field theory.

III. The Algebraic, Ladder Operator Method

First we write

$$\frac{1}{2m}\left[\hat{p}^2 + (m\omega\hat{x})^2\right]\psi = E\psi.$$

While a sum of squares can't be factored over the real numbers, in the complex domain we have

$$z^2 + w^2 = (iz+w)(-iz+w).$$

However, for the Hamiltonian $\hat{H}$ we have the added complication that $\hat{p}$ and $\hat{x}$ are operators. Let's try anyway; we define

$$\hat{a}_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}\left(\mp i\hat{p} + m\omega\hat{x}\right),$$

where the leading factor is purely to make subsequent formulas nicer. Notice that the labels $\pm$ on $\hat{a}$ are not mnemonic for the signs in their definitions. Why these labels are present will be clearer as we proceed.

Using these definitions we can compute

$$\begin{aligned}\hat{a}_-\hat{a}_+ &= \frac{1}{2\hbar m\omega}(i\hat{p} + m\omega\hat{x})(-i\hat{p} + m\omega\hat{x})\\ &= \frac{1}{2\hbar m\omega}\left[\hat{p}^2 + (m\omega\hat{x})^2 - im\omega(\hat{x}\hat{p} - \hat{p}\hat{x})\right].\end{aligned}$$

Notice that in the last equality the order of operators matters! (This would also be true for matrices.) We call

$$[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A},$$

the commutator of $\hat{A}$ and $\hat{B}$. So,

$$\hat{a}_-\hat{a}_+ = \frac{1}{2\hbar m\omega}\left[\hat{p}^2 + (m\omega\hat{x})^2\right] - \frac{i}{2\hbar}[\hat{x}, \hat{p}].$$

Warning: Don't treat operators like #'s, it's too easy to make mistakes. To do it carefully, always act on a test function, say $f(x)$.

For example,

$$\begin{aligned}[\hat{x},\hat{p}]\,f(x) &= (\hat{x}\hat{p}-\hat{p}\hat{x})f(x)\\ &= \hat{x}\frac{\hbar}{i}\frac{df}{dx}-\hat{p}\,(xf(x))\\ &= x\frac{\hbar}{i}\frac{df}{dx}-\frac{\hbar}{i}\frac{d}{dx}(xf)\\ &= -\frac{\hbar}{i}f = i\hbar f.\end{aligned}$$

Then,

$$\boxed{[\hat{x},\hat{p}] = i\hbar.}$$

This result is known as the "canonical commutation relation" for $\hat{x}$ and $\hat{p}$.

Returning to $\hat{H}$, notice that we can now write

$$\hat{a}_-\hat{a}_+ = \frac{1}{\hbar\omega}\hat{H}+\frac{1}{2}$$

or

$$\hat{H} = \hbar\omega\left(\hat{a}_-\hat{a}_+ - \frac{1}{2}\right).$$

Order matters—if we'd computed

$$\hat{a}_+\hat{a}_- = \frac{1}{\hbar\omega}\hat{H}-\frac{1}{2}.$$

In other words,

$$[\hat{a}_-,\hat{a}_+] = 1.$$

Then, in the other ordering

$$\hat{H} = \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right).$$

Finally, the Schrödinger equation becomes (with both orderings)

$$\hbar\omega\left(\hat{a}_\pm\hat{a}_\mp \pm \frac{1}{2}\right)\psi = E\psi.$$

IV. The Prestige

Here's the magic. Suppose ψ is a solution with energy E, then $\hat{a}\psi$ is a new solution with energy $(E+\hbar\omega)$, that is,

$$\hat{H}(\hat{a}_+\psi) = (E+\hbar\omega)(\hat{a}_+\psi).$$

Proof:

$$
\begin{aligned}
\hat{H}(\hat{a}_+\psi) &= \hbar\omega\left(\hat{a}_+\hat{a}_- + \frac{1}{2}\right)(\hat{a}_+\psi) \\
&= \hbar\omega\left(\hat{a}_+\hat{a}_-\hat{a}_+ + \frac{1}{2}\hat{a}_+\right)\psi \\
&= \hbar\omega\hat{a}_+\left(\hat{a}_-\hat{a}_+ + \frac{1}{2}\right)\psi \\
&= \hbar\omega\hat{a}_+\left(\hat{a}_+\hat{a}_- + 1 + \frac{1}{2}\right)\psi \\
&= \hat{a}_+\left(\hat{H} + \hbar\omega\right)\psi \\
&= \hat{a}_+\left(E + \hbar\omega\right)\psi \\
&= \left(E + \hbar\omega\right)\hat{a}_+\psi. \qquad \blacksquare
\end{aligned}
$$

You may not be surprised that

$$\hat{H}(\hat{a}_-\psi) = (E - \hbar\omega)(\hat{a}_-\psi).$$

This is a remarkable tool. The $\hat{a}$'s are ladder operators with $\hat{a}_+$ the 'raising operator' and $\hat{a}_-$ the 'lowering operator'. Is the ladder infinite? No!

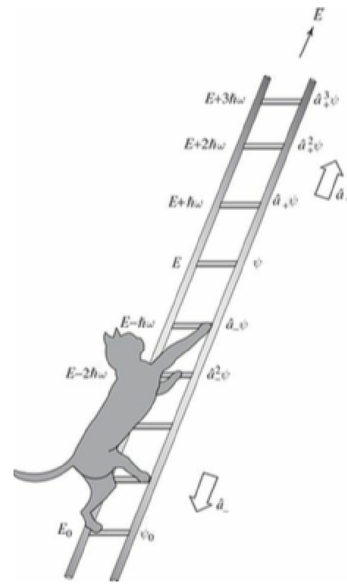

Figure 2: The ladder operators move you up and down the ladder of states for the harmonic oscillator.